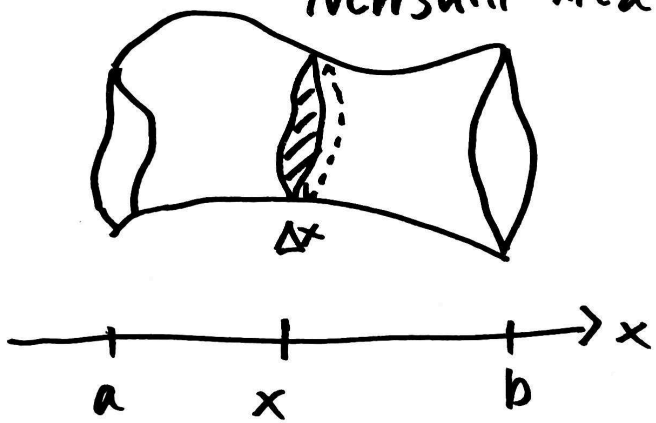
22 april
26

15F Volum som integraler

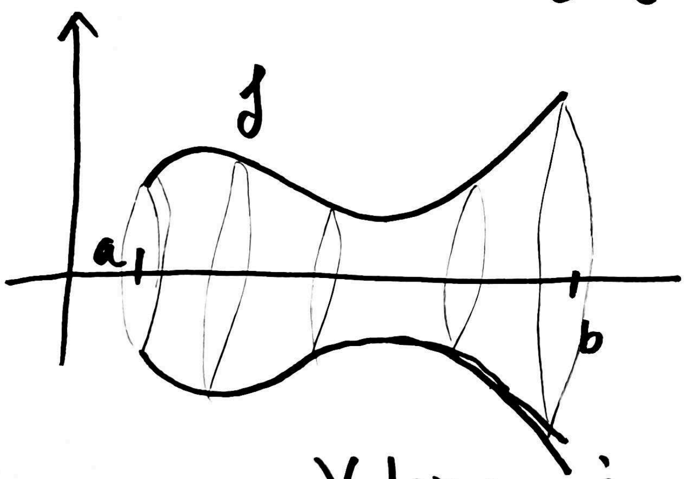
tverrsnitt med areal $A(x)$



$$V \sim \sum_i A(x_i) \cdot \Delta x_i$$

$$V = \int_a^b A(x)\,dx$$

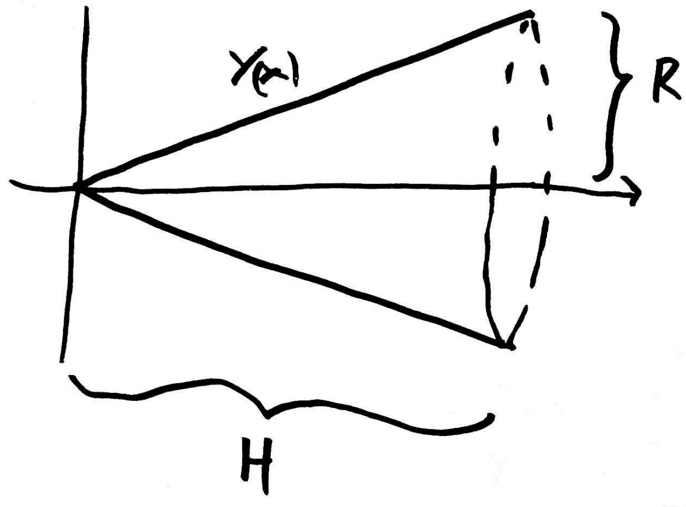
omdreiningslegemer



tverrsnittet i x er en disk med radius $|f(x)|$ og arealet $\pi(f(x))^2$

Volum: $V = \int_a^b \pi(f(x))^2\,dx$

Kjegle



høyde H

radius til bunnflaten R

$$y(x) = \frac{R}{H}x$$

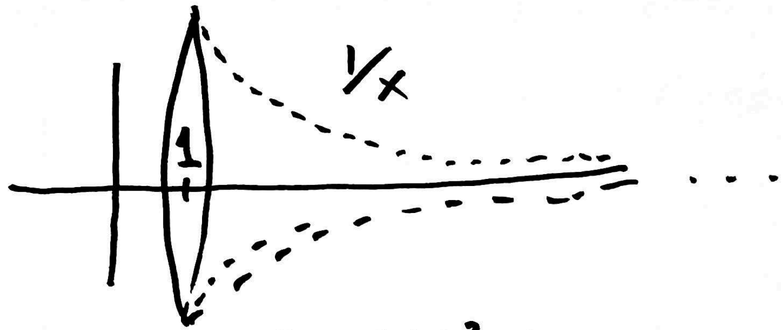
$$V = \int_0^H \pi y^2 \, dx$$

$$= \pi \int_0^H \left(\frac{R}{H}\right)^2 x^2 \, dx = \pi \left(\frac{R}{H}\right)^2 \frac{x^3}{3}\Big|_0^H$$

$$V = \pi \left(\frac{R}{H}\right)^2 \left(\frac{H^3 - 0}{3} \right) = \frac{\pi}{3} R^2 \frac{H^3}{H^2}$$

$$V = \underline{\frac{\pi R^2 H}{3}}$$

Gabriels horn



$$V = \int_1^{\infty} \pi\left(\frac{1}{x}\right)^2 dx$$

uegentlig integral

$$= \lim_{n\to\infty} \int_1^{n} \pi\left(\frac{1}{x}\right)^2 dx$$

$$= \lim_{n\to\infty} \pi\int_1^{n} x^{-2} dx = \lim_{n\to\infty} \pi\left(\frac{-1}{x}\right)\Big|_1^n$$

$$V = \lim_{n\to\infty} \pi\left(\frac{-1}{n} - (-1)\right) = \lim_{n\to\infty} \pi\left(1 - \frac{1}{n}\right)$$

$$\underline{V = \pi}$$

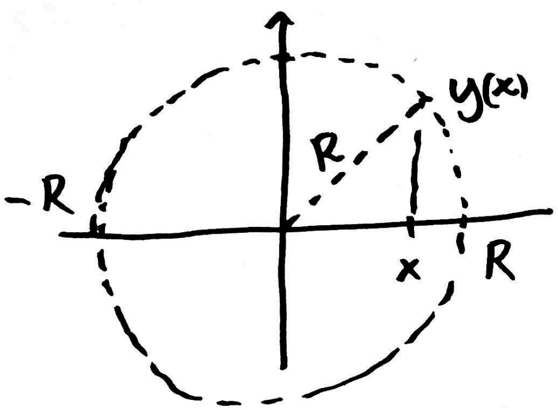
overflate $> \frac{2\pi}{x}\Delta x$

overflaten er derimot uendelig.

$$O \geq \int_1^{\infty} \frac{2\pi}{x} dx = \lim_{n\to\infty} 2\pi \ln x \Big|_1^n$$

$$= 2\pi \lim_{n\to\infty} \ln(n)$$ går mot uendelig.

Kulen



Ved pythagoras

$$y(x) = \sqrt{R^2 - x^2}.$$

$$y^2 = R^2 - x^2$$

$$V_{kule} = \int_{-R}^{R} \pi y^2 \, dx$$

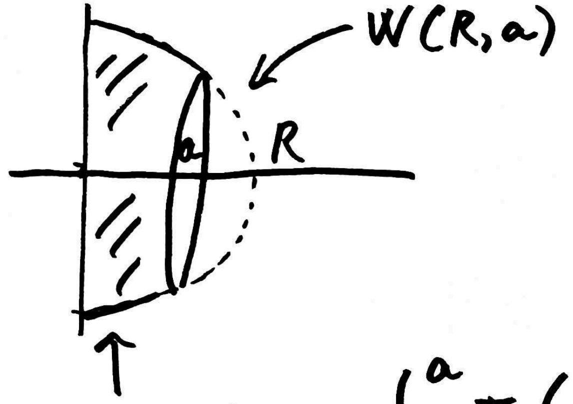
$$= \pi \int_{-R}^{R} R^2 - x^2 \, dx$$

$$V_{kule} = \pi \left(R^2 x - \frac{x^3}{3} \right) \Big|_{-R}^{R} = \pi \left(\underbrace{\left(R^3 - \frac{R^3}{3} \right)} - \left(-R^3 - \frac{(-R)^3}{3} \right) \right)$$

$$\pi \left(\frac{2R^3}{3} - \frac{-2R^3}{3} \right)$$

$$\underline{V_{kule} = \frac{4\pi R^3}{3}}.$$

Avkuttet kule:

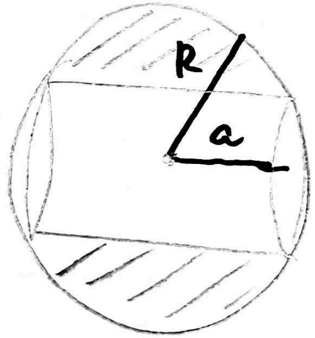


$$V(R,a) = \int_0^a \pi (R^2 - x^2)\, dx$$

$$= \pi \left(R^2 x - \frac{x^3}{3} \right) \Big|_0^a = \pi \left(R^2 a - \frac{a^3}{3} \right)$$

$$V(R,a) = \underline{\pi \left(R^2 a - \frac{a^3}{3} \right)}$$

$$W(R,a) = \frac{2\pi R^3}{3} - V(R,a) = \underline{\frac{2\pi R^3}{3} - \pi R^2 a + \frac{a^3}{3}}$$



Volum til perlen $VP(R,a)$

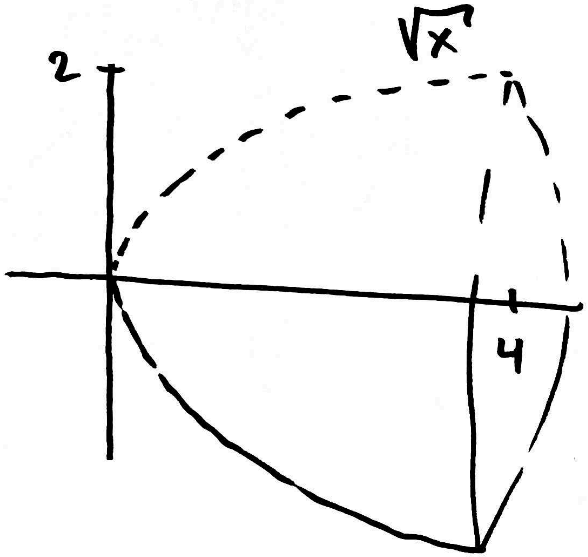
Ved forrige eksempel

$VP(R,a) = \underset{\substack{\uparrow \\ \text{to halvdele}}}{2V(R,a)}$ − Volum sylinder med radius $\sqrt{R^2 - a^2}$ og høyde $2a$

$$VP(R,a) = 2\pi\left(R^2 a - \frac{a^3}{3}\right) - \pi(R^2 - a^2)\cdot 2a$$
$$= 2\pi\left(R^2 a - \frac{a^3}{3} - R^2 a + a^3\right)$$
$$= 2\pi\left(\frac{2a^3}{3}\right) = \underline{\frac{4\pi a^3}{3}}$$

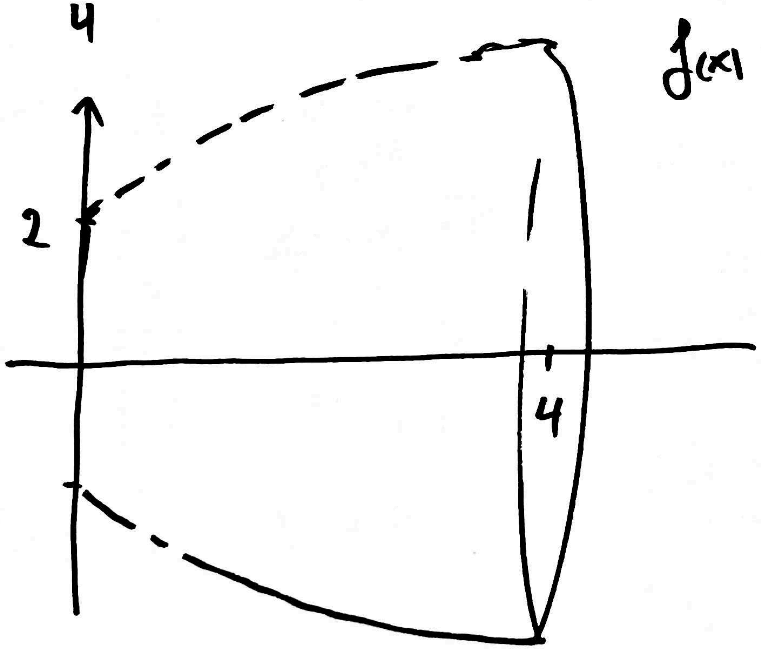
Oppg.

a)



Hva er volumet?

b)

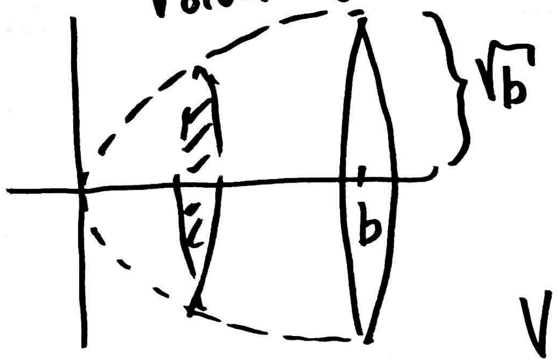


$f(x) = 2+\sqrt{x}$

Hva er volumet?

a)

Volum $V(b)$



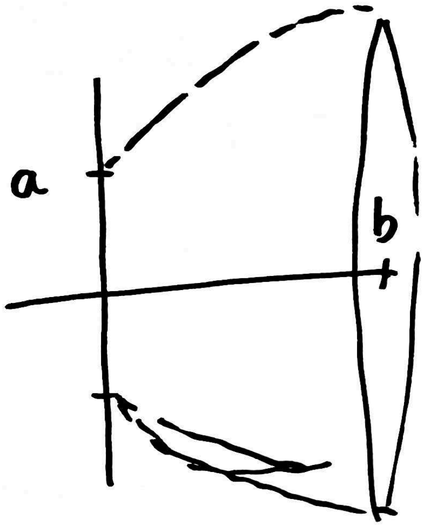
Volum kjegle $\qquad$ Volum sylinder

$$\frac{1}{3}\pi b^2 < V(b) < \pi b^2$$

$$V(b) = \int_0^b \pi(\sqrt{x})^2\,dx = \pi\int_0^b x\,dx$$

$$= \pi \frac{x^2}{2}\Big|_0^b = \underline{\frac{\pi}{2}b^2}$$

halvparten av volumet til sylinderen (den minste) som inneholder legemet.



$a = 2$
$b = 4$
$y(x) = a + \sqrt{x}$

$$V(a,b) = \int_0^b \pi y^2\,dx = \int_0^b \pi(a+\sqrt{x})^2\,dx$$

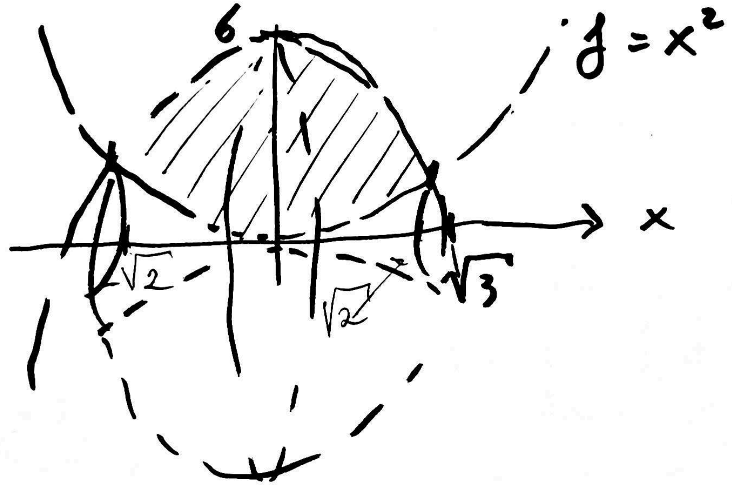
$$\pi\int_0^b a^2 + 2a\overbrace{\sqrt{x}}^{x^{1/2}} + x\,dx = \pi\left(a^2x + 2a\frac{x^{3/2}}{3/2} + \frac{x^2}{2}\right)\Big|_0^b$$

$$V(a,b) = \underline{\pi\left(a^2b + \frac{4a}{3}b^{3/2} + \frac{b^2}{2}\right)}$$

$$V(2,4) = \pi\left(2^2\cdot 4 + \frac{4\cdot 2}{3}4^{3/2} + \frac{4^2}{2}\right) = \pi\left(16 + \frac{64}{3} + \frac{16}{2}\right)$$

$$= \underline{\pi(24 + 64/3)}$$

Funksjone $f(x) = x^2$ og $g(x) = 6-2x^2$ avgrenser et (endelig) område. Roter dette området om x-aksen og regn ut volumet.



$f(x) = g(x)$

$x^2 = 6-2x^2$ og $x = \pm\sqrt{2} \sim \pm 1.41$

$3x^2 = 6$ så $x^2 = 2$

$$\int_{-\sqrt{2}}^{\sqrt{2}} \pi g^2(x) - \pi f^2(x)\, dx = \pi \int_{-\sqrt{2}}^{\sqrt{2}} (6-2x^2)^2 - (x^2)^2\, dx$$

symmetrisk om y-akse.

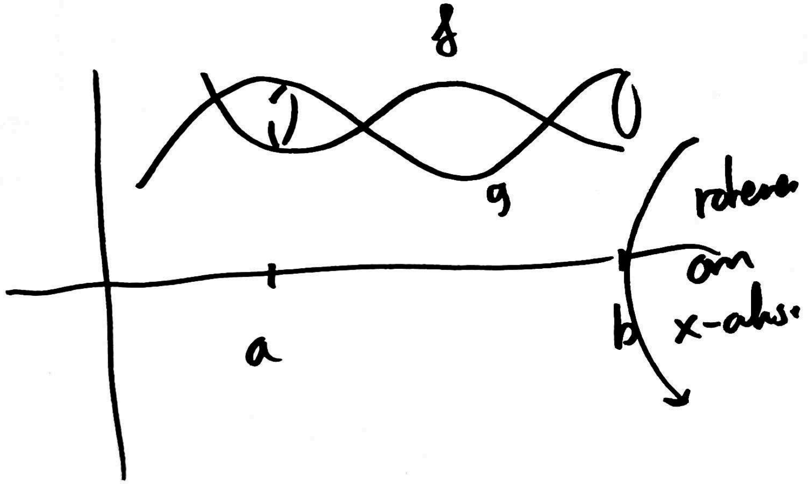
$$= \pi \int_{-\sqrt{2}}^{\sqrt{2}} 36 - 24x^2 + 3x^4\, dx$$

$$= \cancel{2}\pi \int_{\cancel{0}}^{\sqrt{2}} 36 - 24x^2 + 3x^4\, dx = 2\pi \left(36x - 8x^3 + \frac{3}{5}x^5 \right)\Big|_0^{\sqrt{2}}$$

$$V = 2\pi\left(36\sqrt{2} - 8\cdot 2\sqrt{2} - \frac{3}{5}\,2\cdot 2\sqrt{2}\right)$$

$$= 2\pi\sqrt{2}\left(36 - 16 - \frac{12}{5}\right) = 2\pi\sqrt{2}\left(20 - \frac{12}{5}\right)$$

$$= \frac{2\pi\sqrt{2}}{5}(100-12) = \frac{2\pi\sqrt{2}\cdot 88}{100}$$

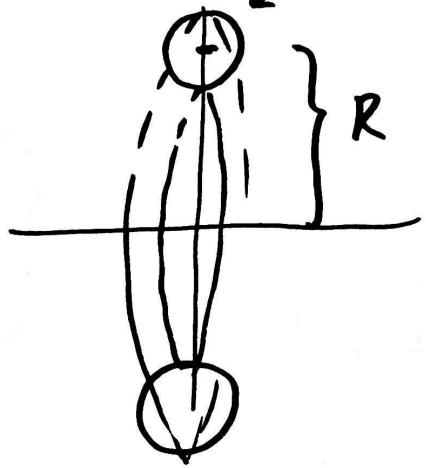


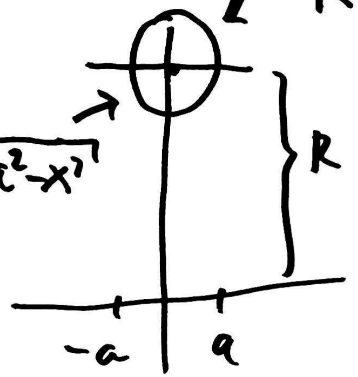
$$V = \int_a^b |f^2 - g^2|\,dx$$

Torus

(smultring)

$0 < a < R$



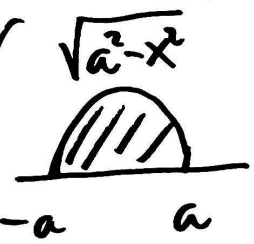


$$VT(R,a) = \int_{-a}^{a} \pi\left(R+\sqrt{a^2-x^2}\right)^2 - \pi\left(R-\sqrt{a^2-x^2}\right)^2 dx$$

$$= \pi\int_{-a}^{a} 4R\sqrt{a^2-x^2}\,dx = 4R\pi\cdot\frac{\pi a^2}{2} = \underline{2\pi^2 R a^2}$$

$$VT(R,a) = \underline{(2\pi R)(\pi a^2)}$$

(



$$\int_{-a}^{a} \sqrt{a^2-x^2}\,dx = \frac{1}{2}\pi a^2$$

halvparten av arealet
til en disk (sirkel) med radius a)

Øving

15.106 b) $\int e^x \sin x \, dx$

Vi har tidlige regnet ut $\int e^{ax} \sin bx \, dx$.

Dette er tilfellet med $a=1$ og $b=1$

$$\int e^x \sin x \, dx = \frac{1}{2}\left(e^x \sin x - e^x \cos x\right)$$ (notatene fra mandag 20.04)

15.116 Tilsvarende eksempel fra forelesning.

15.101 $$\int \underbrace{x^r}_{v'} \underbrace{\ln x}_{u} \, dx = \frac{x^{r+1}}{r+1} \ln x - \int \frac{x^r}{r+1} \, dx$$ $r \neq -1$

$$= \frac{x^{r+1}}{r+1} \ln x - \frac{x^{r+1}}{(r+1)^2} + C$$

$r=-1$ $$\int \underbrace{\frac{1}{x}}_{(\ln x)'} \ln x \, dx = \int u \, du = \frac{u^2}{2} + C$$ $u = \ln x$

$$= \underline{\frac{1}{2}(\ln x)^2 + C}$$

$$\int \frac{1}{x}(\ln x)^r \, dx = \begin{cases} \dfrac{(\ln x)^{r+1}}{r+1} + c & r \neq -1 \\ \ln|\ln x| + c \end{cases}$$

d) $\int_1^3 x^2 \ln x \, dx = \frac{x^3}{3}\ln x - \frac{x^3}{9} + c \Big|_1^3$

$= \frac{3^3}{3}\ln 3 - \frac{3^3}{9} - \Big(\underbrace{\frac{1}{3}\ln 1}_{0} - \frac{1}{9}\Big)$

$= \underline{9\ln 3 - 3 + \frac{1}{9}}$

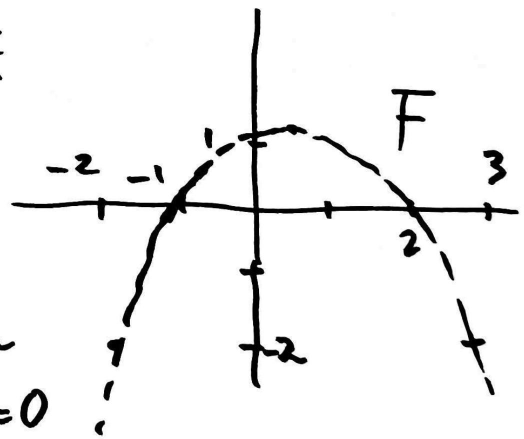
15.83 $f(x) = F'(x)$ F er en antiderivert til $f(x)$

a) $\int_{-2}^{2} f(x)\,dx =$

$F\Big|_{-2}^{2} = \underbrace{F(2)}_{} - F(-2)$

$= 0 - (-2)$

$= \underline{2}$



$F(-2) = -2$
$F(3) = -2$

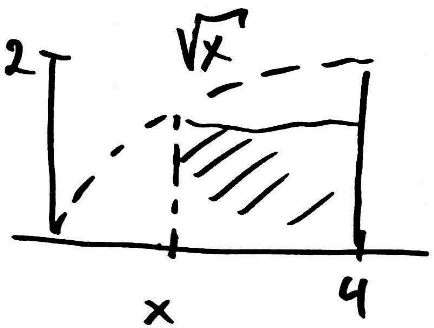
Finn k s.a

b) $\int_{-2}^{k} f(x)\,dx = 0$

$F(k) - F(-2) = 0$

Løsning $k = -2$ og $k = 3$.

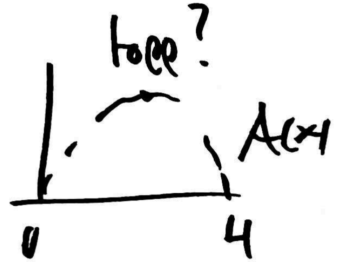
Ex 2025
1 des



Når er arealet størst?

$$A(x) = \underset{\text{høyde}}{\sqrt{x}} \cdot \underset{\text{bredde}}{(4-x)}$$

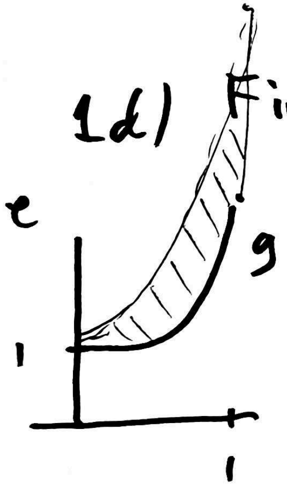
$$A(x) = \underline{4\sqrt{x} - x^{3/2}}$$



$$A'(x) = 0 \quad \dots$$

Eksamen
9. aug 21

1d) Finn $\underline{\text{arealet}}$ begrenset av $f(x) = (x+1)e^{x^2}$, $g(x) = e^{x^2}$ og linjene $x=0$, $x=1$.



Vi har $f(x) = (x+1)\cdot g(x)$

$\left(\int_0^1 e^{x^2}\,dx \right.$ Finner vi ikke en elementær antiderivert... $\left. \right)$

$f \geq g$ i $[0,1]$

$$A = \int_0^1 |f-g|\,dx = \int_0^1 f-g\,dx = \int_0^1 x e^{x^2}\,dx$$

substitusjon
$u = x^2$
$du = 2x\,dx$

$$= \int_0^{1^2} e^u \frac{du}{2} = \frac{1}{2} e^{x^2}\Big|_0^1 = \underline{\frac{1}{2}(e-1)}$$

15.66 $\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx$ $\quad u = \cos x$, $u' = -\sin x$

$= \int \frac{-1}{u} du = -\ln|u| + c$

$\int \tan x \, dx = \underline{-\ln|\cos x| + c}$

$a \neq 0$

$\int \tan(ax) dx$ $\quad u = ax$, $u' = a$, $du = a\,dx$, $\frac{1}{a} du = dx$

$= \int \tan(u) \frac{1}{a} du$

$= \frac{1}{a} (-\ln|\cos(u)| + c$

$= \underline{\frac{-1}{a} \ln|\cos(ax)| + C}$

$(\tan x)' = 1 + \tan^2 x$

b) $\int \tan^2 x \, dx$

$= \int (\tan x)' - 1 \, dx = \underline{\tan(x) - x + c}$