20 april
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15E Delvis integrasjon

Produktregelen for derivasjon: $(F(x)G(x))' = F'(x) \cdot G(x) + F(x) \cdot G'(x)$

La $f(x) = F'(x)$ og $g(x) = G'(x)$

$$(F(x) \cdot G(x))' = f(x)G(x) + F(x) \cdot g(x)$$

så $\int f(x)G(x) + F(x)g(x)\,dx = F(x)G(x) + c$

Eksempel $F(x) = x$, $f(x) = 1$
$G(x) = \sin x$, $g(x) = \cos x$

$$\int \sin x + x\cos x\,dx = x\sin x + c$$

$$\int f \cdot G \, dx + \int F \cdot g \, dx = F \cdot G + c$$

$$\boxed{\int f \cdot G \, dx = F \cdot G - \int F \cdot g \, dx}$$

Delvis integrasjon

Eks

$$\int \overset{G}{x} \overset{f}{\cos x} \, dx = x(\sin x) - \underbrace{\int 1 \cdot \sin x \, dx}_{-\cos x + c}$$

$$\int x \cos x \, dx = \underline{x \sin x + \cos x + C}$$

* $\int \overset{u}{x}\,\overset{v'}{e^x}\,dx = x \cdot e^x - \int \overset{u'}{1} \cdot \overset{v}{e^x}\,dx$

$= \underline{x e^x - e^x + c}$

* $\int \overbrace{4x}^{f}\,\overbrace{e^{3x-1}}^{g'}\,dx$

$= 4x\,\frac{1}{3}e^{3x-1} - \int 4 \cdot \frac{1}{3} e^{3x-1}\,dx$

$= \frac{4x}{3} e^{3x-1} - \frac{4}{3}\int e^{3x-1}\,dx$

$= \frac{4x}{3} e^{3x-1} - \frac{4}{3}\left(\frac{1}{3} e^{3x-1}\right) + c$

$= \frac{4x}{3} e^{3x-1} - \frac{4}{9} e^{3x-1} + c$

$= \underline{\frac{4}{9}(3x-1)\,e^{3x-1} + c}$

$f = 4x$

$f' = 4$

$g' = e^{3x-1}$

$g = \frac{1}{3} e^{3x-1}$

$$* \int x^3 e^{x^2} dx$$

$$= \int (2x) \cdot \frac{1}{2} x^2 e^{x^2} dx$$

$$= \frac{1}{2} \int u e^u \, du$$

$$= \frac{1}{2} \left(u e^u - e^u \right) + C$$

$$= \underline{\frac{1}{2} (x^2 - 1) e^{x^2} + C}$$

$$x^2 = u$$

$$\frac{du}{dx} = u' = 2x$$

$$du = u' dx = 2x \, dx$$

$$du = \frac{du}{dx} dx$$

(som ovenfor)

Minner om substitusjon

$$F'(u) = f(u)$$

kjerneregelen

$$\frac{d}{dx}\left(F(u(x))\right) = F'(u(x)) \cdot u'(x)$$

$$= f(u) \cdot u'$$

Så $\int f(u(x)) \, u'(x) \, \underline{dx}$

$$= F(u(x)) + C = \int f(u) \, \underline{du}$$

$$\int \underset{u}{x^2}\, \underset{v'}{e^x}\, dx \qquad v = e^x, \quad v' = e^x$$

$$\overset{\text{delvis int}}{=} x^2 e^x - \int \overset{u'}{2x} \cdot \overset{v}{e^x}\, dx$$

$$= x^2 e^x - 2 \underbrace{\int x e^x\, dx}_{x e^x - e^x} = \underline{(x^2 - 2x + 2) e^x + c}$$

Benytter delvis integrasjon to ganger.

$$\int \overset{u}{x^n}\, \overset{v'}{e^x}\, dx = x^n e^x - \int n x^{n-1} e^x\, dx$$

$$\underline{\int x^n e^x\, dx = x^n e^x - n \int x^{n-1} e^x\, dx}$$

Rekursiv formel

$$\int x^{17} e^x\, dx$$ ... benytte delvis int. 17 ganger

$$= p(x)\, e^x + c$$

$p(x)$ polynom av grad 17.

$$\int \ln|x|\,dx$$

$$(\ln|x|)' = \frac{1}{x}$$

$$u' = 1$$

$$u = x$$

$$= \int \underbrace{1}_{u'} \cdot \underbrace{\ln|x|}_{v}\,dx$$

$$\overset{\text{delvis}}{=} x \cdot \ln|x| - \int \underbrace{x \cdot \frac{1}{x}}_{1}\,dx$$

$$\int \ln|x|\,dx = \underline{x\ln|x| - x + c}$$

$$* \quad \int \underbrace{x^4}_{u'} \underbrace{\ln|x|}_{v}\,dx = \underbrace{\frac{x^5}{5}}_{u} \underbrace{\ln|x|}_{v} - \int \frac{x^5}{5} \cdot \frac{1}{x}\,dx$$

$$= \frac{x^5}{5}\ln|x| - \frac{1}{5}\int x^4\,dx = \underline{\frac{x^5}{5}\ln|x| - \frac{1}{25}x^5 + c}$$

$$\int x^5 \ln(3|x|^7)\,dx$$

$$= \int x^5 (\ln 3 + \ln(|x|^7))\,dx$$

$$= \int x^5 (\ln 3 + 7\ln|x|)\,dx$$

$$= \ln(3) \int x^5\,dx + 7\int \overset{u'}{x^5}\,\overset{v}{\ln|x|}\,dx$$

$$= \frac{\ln(3)}{6} x^6 + 7\left(\frac{x^6}{6} \ln|x| - \int \underbrace{\overset{u}{\frac{x^6}{6}} \cdot \overset{v'}{\frac{1}{x}}}_{\frac{1}{6}x^5}\,dx \right)$$

$$= \frac{\ln 3}{6} x^6 + \frac{7}{6} x^6 \ln|x| - \frac{7}{36} x^6 + C$$

$$= \underline{\frac{x^6}{36}\left(6\ln 3 - 7 + 42\ln|x| \right)}$$

Finn $\int \underbrace{-3x}_{u} \underbrace{\sin(\pi x)}_{v'} dx$

En antiderivert til $\sin(\pi x)$ er $v = \frac{-\cos(\pi x)}{\pi}$.

delvis int

$u = -3x$

$u' = -3$

$$= -3x \frac{-\cos(\pi x)}{\pi} - \int \overset{u'}{-3} \cdot \frac{-\cos(\pi x)}{\pi} dx$$

$$= \frac{3}{\pi} x \cos(\pi x) - \frac{3}{\pi} \underbrace{\int \cos(\pi x)\, dx}_{\frac{\sin(\pi x)}{\pi}} + c$$

$$\int -3x \sin(\pi x)\, dx = \frac{3}{\pi} x \cos(\pi x) - \frac{3}{\pi^2} \sin(\pi x) + c$$

$$\int \overbrace{e^{ax}}^{v'} \overbrace{\sin bx}^{u}\, dx = u \cdot v - \int v \cdot u'\, dx$$

$v = \frac{1}{a} e^{ax}$

$u' = b\cos bx$

$w' = -b\sin bx$

$$= \frac{1}{a} e^{ax} \sin bx - \int \frac{1}{a} e^{ax} b \cos bx\, dx$$

$$= \frac{1}{a} e^{ax} \sin bx - \frac{b}{a} \int \underbrace{e^{ax}}_{v'} \underbrace{\cos bx}_{w}\, dx$$

$$= \frac{1}{a} e^{ax} \sin bx - \frac{b}{a}\left(\frac{1}{a} e^{ax} \cos bx - \int \frac{1}{a} e^{ax}(-b\sin bx)\, dx \right)$$

$$= \frac{1}{a} e^{ax} \sin bx - \frac{b}{a^2} e^{ax} \cos bx - \frac{b^2}{a^2} \int e^{ax} \sin bx\, dx$$

integralet vi startet med!

Løser ut for integralet

$$\left(1 + \frac{b^2}{a^2}\right) \int e^{ax} \sin bx\, dx = \frac{1}{a} e^{ax} \sin bx - \frac{b}{a^2} e^{ax} \cos bx + C$$

$$\int e^{ax} \sin bx\, dx = \frac{1}{a^2+b^2}\left(a e^{ax} \sin bx - b e^{ax} \cos bx\right) + C$$

$$\int e^{-2x}\sin(\pi x)\,dx = \frac{1}{4+\pi^2}\left(-2e^{-2x}\sin \pi x - \pi e^{-2x}\cos \pi x\right)+c$$

$a=-2$

$b=\pi$

u' bør være "enklere enn u"

$$\int u \cdot v'\,dx = u\cdot v - \int u'\cdot v\,dx$$

må klare å finne antiderivert til v'.

et enklere integral

Delbrøksoppspalting.

$$\frac{1}{x^2-x} = \frac{1}{(x-1)x} = \frac{1}{x-1} - \frac{1}{x} \quad \left(\text{fordi } \frac{x}{(x-1)\cdot x} - \frac{(x-1)}{x(x-1)} = \frac{x-(x-1)}{x(x-1)} = \frac{1}{x(x-1)} \right)$$

$$\int \frac{1}{x^2-x}\,dx = \int \frac{1}{x-1} - \frac{1}{x}\,dx$$

$$= \ln|x-1| - \ln|x| + C$$

$$\int \frac{1}{x^2-x}\,dx = \underline{\ln\left|\frac{x-1}{x}\right| + C}$$

$$* \quad \int \frac{1}{x^2-9}\,dx = \int \frac{1}{(x-3)(x+3)}\,dx$$

delbrøksoppspalting: $\frac{1}{(x-3)(x+3)} = \frac{A}{x-3} + \frac{B}{x+3}$.

$$1 = A(x+3) + B(x-3)$$

setze $x = -3$: $\quad 1 = A \cdot 0 + B(-6)$

$B = -1/6$

$x = 3$: $\quad 1 = 6 \cdot A + B \cdot 0$

$A = 1/6$

$$\int \frac{1}{x^2 - 9}\, dx = \int \frac{1/6}{x-3} + \frac{-1/6}{x+3}\, dx$$

$$= \frac{1}{6} \ln|x-3| - \frac{1}{6} \ln|x+3| + C$$

$$= \underline{\frac{1}{6} \ln\left|\frac{x-3}{x+3}\right| + C}$$

Øving

Eksamensoppgave

2023 $\int 2x\ln|x|\,dx = 2\int \underbrace{x}_{v'}\underbrace{\ln|x|}_{u}\,dx = 2\left(\frac{x^2}{2}\ln|x| - \int \underbrace{\frac{x^2}{2}\cdot\frac{1}{x}}_{x/2}\,dx\right.$

$= 2\left(\frac{x^2}{2}\ln|x| - \frac{x^2}{4}\right) + C = \underline{x^2\ln|x| - \frac{x^2}{2} + C}$

* $\int 3x\ln|(2x)^5|\,dx = \int 3x\cdot 5\ln|2x|\,dx$

$= 15\int (x\ln 2 + x\ln|x|)\,dx$

$= 15\left(\frac{x^2}{2}\ln 2 + \left(\frac{x^2}{2}\ln|x| - \frac{x^2}{4}\right)\right) + C$ ved delvis integrasjon

$= \underline{15\left(\frac{\ln 2}{2}x^2 + \frac{x^2}{2}\ln|x| - \frac{x^2}{4}\right) + C}$

Eksamen mai 2025

$$\int_0^1 \underbrace{(2x+1)}_{u}\underbrace{e^x}_{v'}\,dx = (2x+1)e^x \Big|_0^1 - \int_0^1 2\cdot e^x\,dx$$

$$= \left((2x+1)e^x - 2e^x\right)\Big|_0^1$$

$$= 3e^1 - 2e^1 - (e^0 - 2e^0)$$

$$= \underline{e+1} \sim 3.71$$

$$\int_a^b u'\cdot v\,dx = u\cdot v\Big|_a^b - \int_a^b u\cdot v'\,dx$$

Delvis integrasjon for bestemte integraler

Oppg 15.94 $\int \sin x \cos x \, dx$ Løser integralet på 3 ulike måter.

1. Delvis integrasjon: $\int \overset{u}{\sin x} \, \overset{v'}{\cos x} \, dx = \sin x \cdot \sin x - \int \overset{u'}{\cos x} \, \overset{v}{\sin x} \, dx$

Flytte over og løse for integral

$\int \sin x \cos x \, dx = \frac{1}{2} \sin^2 x + C$

2. Substitusjon: $\int \overset{u}{\sin x} \, \overset{u'}{\cos x} \, dx = \int u \, du = \frac{1}{2} u^2 + C = \frac{1}{2} \sin^2 x + C$

3. Trig. identitet: $\int \sin x \cos x \, dx = \int \frac{1}{2} \sin 2x \, dx = \frac{1}{2}\left(-\frac{\cos 2x}{2}\right) + C$

$= \frac{-1}{4}(\cos^2 x - \sin^2 x) + C \overset{\text{Pytagoras}}{=} \frac{-1}{4}(1 - \sin^2 x - \sin^2 x) + C$

$= \frac{1}{2} \sin^2 x +$ konstant.

15.95 (ex2017) a) $\int \frac{2 + \sqrt{x}}{x\sqrt{x}} \, dx = \int \frac{2}{x^{3/2}} + \frac{1}{x} \, dx = \int 2x^{-3/2} + \frac{1}{x} \, dx$

$= 2 \cdot \frac{x^{-1/2}}{-1/2} + \ln|x| + C = \frac{-4}{\sqrt{x}} + \ln|x| + C$

b) $\int_0^{\infty} x e^{1-x^2} \, dx$, finner antiderivert $\int x e^{1-x^2} \, dx$

$u = 1 - x^2$
$u' = -2x$
$du = -2x \, dx$
$-\frac{1}{2} du = x \, dx$

$\int e^u \left(\frac{-1}{2}\right) du = \frac{-1}{2} e^u + C = \frac{-1}{2} e^{1-x^2}$

$$\int_0^\infty x e^{1-x^2} dx = \lim_{n\to\infty} \int_0^n x e^{1-x^2} dx$$

$$= \lim_{n\to\infty} \left. \frac{-1}{2} e^{-1-x^2} \right|_0^n = \lim_{n\to\infty} \frac{-1}{2} e^{-1-n^2} + \frac{1}{2} e^{-1}$$

$$= \frac{1}{2} e^{-1} = \underline{\frac{1}{2e}}$$