15.04
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15D og E Substitusjon

$$\int (x+3)^2\,dx$$

I $\int x^2+6x+9\,dx = \frac{x^3}{3} + 3x^2 + 9x + C$

II $\left((x+3)^3\right)' \underset{\text{kjerneregel}}{=} 3(x+3)^2 \underbrace{(x+3)'}_{1} = 3(x+3)^2$, så $\left(\frac{(x+3)^3}{3}\right)' = (x+3)^2$

Derfor er $\int (x+3)^2\,dx = \frac{(x+3)^3}{3} + C$ (ny konstant)

$$\int \frac{4}{x+3}\,dx$$

$$= 4\int \frac{1}{x+3}\,dx = \underline{4\ln|x+3| + C}$$

$\left(\ln|x+3|\right)' = \frac{1}{x+3}\overbrace{(x+3)'}^{1}$
$= \frac{1}{(x+3)}$

$\int (x+3)^{11}\,dx$

Det er mye arbeid å gange ut, og så integrere ...

$$\left((x+3)^{12}\right)' \overset{\text{kjerne}}{=} 12(x+3)^{11}\underbrace{(x+3)'}_{1} = 12(x+3)^{11}$$

Så $\int (x+3)^{11}\,dx = \frac{(x+3)^{12}}{12} + C$

svaret er på en fin form.

Lineær substitusjon $\quad u = ax+b,\ u' = a$

$F'(x) = f(x)$

$$\big(F(\overbrace{ax+b}^{u})\big)' = F'(u)\cdot u'$$
$$= f(ax+b)\cdot a$$

$$\left(\frac{1}{a}F(ax+b)\right)' = f(ax+b)$$

$$\int f(ax+b)\,dx = \frac{1}{a}F(ax+b) + c$$

hvor $F'(x) = f(x)$

$$= \frac{1}{a}\underbrace{\int f(u)\,du}_{F(u) \text{ opp til konstantledd.}} + c$$

$5x-7 = u$

$5 = u'$

$$\int \frac{4}{5x-7}\,dx = 4\int \frac{1}{5x-7}\,dx$$

$$= 4\cdot\frac{1}{5}\int\frac{1}{u}\,du$$

$$= 4\cdot\frac{1}{5}\ln|u| + c$$

$$= \underline{\frac{4}{5}\ln|5x-7|} + c$$

(Generell) substitusjon
Variabelbytte.

$$F'(x) = f(x)$$

$$\left(F(u(x))\right)' \overset{\text{kjerneregelen}}{=} F'(u(x)) \cdot u'(x) = u'(x) f(u(x)) = f(u(x))\, u'(x)$$

$$\int f(u(x)) \cdot u'(x)\, dx = \int f(u)\, du \quad (= F(u) + c)$$

$$\int 4x^3(1+x^4)^7\,dx$$

La $\quad u = 1+x^4$

$u' = 4x^3$

$$\int u'(x)\cdot u^7\,dx = \int u^7 \underbrace{u'(x)\,dx}_{du}$$

$$= \int u^7\,du$$

$$= \frac{u^8}{8} + c$$

$$\int 4x^3(1+x^4)^7\,dx = \frac{(1+x^4)^8}{8} + c$$

$$\int \underbrace{\sin^2 x}_{u^2} \underbrace{\cos x}_{u'} \, dx$$

$u = \sin x$

$u' = \cos x$

$$= \int u^2 \, du = \frac{u^3}{3} + C$$

$$= \underline{\frac{\sin^3 x}{3} + C}$$

$I = \int \sin^3 x \, dx$

integranden er ikke på formen $u'(x) f(u)$.

Benytter identiteten

$\sin^2 x + \cos^2 x = 1$

$\sin^3 x = \sin x (1 - \cos^2 x)$

$I = \int \sin x - \sin x \cdot \cos^2 x \, dx$

$$I = \int \sin x \, dx + \int \underset{u'}{(-\sin x)} \cdot \underset{u^2}{(\cos x)^2} \, dx$$

$$= -\cos x + \int u^2 \, du = -\cos x + \frac{u^3}{3} + c$$

$$\int \sin^3 x \, dx = \underline{-\cos x + \frac{\cos^3 x}{3} + c}$$

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = \int \sin x \cdot \left(\frac{1}{\cos x}\right) dx$$

bemyte $u = \cos x$

$u' = -\sin x$

$$= \int (-u') \cdot \frac{1}{u} \, dx = -\int \frac{1}{u} \, du$$

$$= -\ln|u| + c$$

$$\int \tan x \, dx = \underline{-\ln|\cos x| + c}$$

$\int t\sqrt{t+1}\,dt$

$t+1 = u$ (legger til -1:)

$t = u-1$

$dt = du$ $\left(\frac{du}{dt} = u' = 1\right)$

$\int (u-1)u^{1/2}\,du$

$\int u^{3/2} - u^{1/2}\,du$

$= \frac{u^{5/2}}{5/2} - \frac{u^{3/2}}{3/2} + C$

$= \frac{2}{5}(t+1)^{5/2} - \frac{2}{3}(t+1)^{3/2} + C$

$= \left(\frac{2}{5}(t+1)^2 - \frac{2}{3}(t+1)\right)\sqrt{t+1} + C$

$\int (x^2+1)^2\,dx$

$u = x^2+1$ $u' = 2x$

= mangler en faktor x i integranden.

kan ikke benytte subst.

gange ut

$$\int x^4 + 2x^2 + 1 \, dx = \underline{\frac{x^5}{5} + \frac{2}{3}x^3 + x + C}$$

Substitusjon med Leibniz notasjon

$$\int f(u)\, u' \, dx = \int f(u) \frac{du}{dx} \, dx$$

$$= \int f(u) \, du$$

$$u'(x) = \frac{du}{dx}$$

$$u'(x) dx = du$$

$$\frac{du}{dx} dx = du$$

$$\int \frac{1}{x} \ln|x^3|\,dx = \int \frac{1}{x} \ln(|x|)^3\,dx$$

$$= \int \frac{1}{x} \cdot 3\ln|x|\,dx$$

$$(\underbrace{\ln|x|}_{u})' = \underbrace{\frac{1}{x}}_{u'}$$

$$= 3\int \underbrace{\frac{1}{x}}_{u'} \cdot \underbrace{\ln|x|}_{u}\,dx$$

$$3\int u\,du = 3\,\frac{u^2}{2} + c = \underline{\frac{3}{2}(\ln|x|)^2 + c}$$

Oppg. a) $\int e^{3x-1}\,dx$

$u = 3x-1 \qquad u' = 3$

$$\int \frac{1}{3} \cdot \underbrace{3}_{u'}\, e^u\,dx = \frac{1}{3}\int e^u\,du$$

$$= \underline{\frac{1}{3} e^{3x-1} + c}$$

b) $\int x e^{-x^2}\,dx$

Prøver med $u = -x^2$
$u' = -2x$

$$\int \frac{-1}{2}\underbrace{(-2x)}_{u'}\, e^u\,dx = \frac{-1}{2}\int e^u\,du$$

$$= \underline{\frac{-1}{2} e^{-x^2} + c}$$

$$I = \int_a^b \underbrace{u'(x) f(u(x))}_{(F(u(x)))'} dx \quad \text{hvor } F'(x) = f(x).$$

Ved fundamentalteoremet

$$I = F(u(x)) \Big|_a^b$$

$$= F(u(b)) - F(u(a))$$

$$= \int_{u(a)}^{u(b)} f(x)\, dx$$

$$\int_a^b u'(x) f(u(x))\, dx = \int_{u(a)}^{u(b)} f(u)\, du$$

↑ integralgrensene endres!

$$\int_0^1 (x+2)^2\,dx \qquad u = x+2,\quad u' = 1$$

$$= \int_2^3 u^2\,du \qquad u(0) = 2,\quad u(1) = 3$$

$$= \frac{u^3}{3}\Big|_2^3$$

$$= \frac{1}{3}3^3 - \frac{2^3}{3} = \frac{1}{3}(27-8) = \underline{\underline{\frac{19}{3}}} \left(= 6+\frac{1}{3}\right)$$

Øving

a) $\int \frac{x^3}{2+x^4}\,dx$

b) $\int \cos^5(x)\,dx$ hint: $\cos^2 x + \sin^2 x = 1$

c) $\int \sin^2(x)\,dx$ hint: benytt trig. identiteter

d) $\int x^3(2-x^2)^{13}\,dx$ hint: $u = 2-x^2 \ldots$

bokа 15.70 c) $\int_0^{\pi} \sin\left(\frac{x}{3}\right) - 3\cos(3x)\,dx$

(Kont. eks 4b) $\int_0^1 \frac{x+5}{x^2-2x-3}\,dx$ $x^2-2x-3 = (x-1)^2-4$ $= (x-3)(x+1)$

bruk delbrøksoppspalting (neste uke))

Forslag til løsninger

a) $\int x^3 \cdot \frac{1}{2+x^4}$ $\qquad u = 2+x^4$

$\underline{u' = 4x^3}$

$= \int \frac{1}{4} \cdot \underbrace{4x^3}_{u'} \cdot \frac{1}{u}\,dx = \frac{1}{4}\int u' \frac{1}{u}\,dx = \frac{1}{4}\int \frac{1}{u}\,du$

$= \frac{1}{4}\ln|u| + C = \underline{\frac{1}{4}\ln(2+x^4) + C}$

b) $\int \cos^5 x\,dx = \int \cos(x)\left(\cos^2 x\right)^2 dx = \int \overbrace{\cos x}^{u'} \cdot \overbrace{\left(1-\sin^2 x\right)^2}^{(1-u^2)^2} dx$

La $u = \sin x$
$u' = \cos x$

$= \int (1-u^2)^2\,du = \int 1 - 2u^2 + u^4\,du = \frac{u^5}{5} - \frac{2u^3}{3} + u + C$

$\int \cos^5 x\,dx = \frac{\sin^5 x}{5} - \frac{2\sin^3 x}{3} + \sin x + C$

c) $\sin^2 x = \frac{1}{2}(1-\cos 2x)$ fordi $\cos(2x) = \cos^2 x - \sin^2 x$

$$\int \sin^2 x\, dx = \frac{1}{2}\int 1-\cos(2x)\, dx$$

$$= \frac{1}{2}\left(x - \frac{1}{2}\sin(2x)\right) + C$$

$$= \frac{x}{2} - \frac{1}{4}\sin(2x) + C$$

d) $\int x^3(2-x^2)^{13}\, dx$

$u = 2-x^2$, $x^2 = 2-u$

$u' = -2x$

$x^3 = \underbrace{x^2}_{2-u} \cdot \underbrace{x}_{-\frac{1}{2}u'}$

$$= \int -\frac{1}{2}u'(2-u)u^{13}\, dx$$

$$= \int -\frac{1}{2}(2-u)u^{13}\, du$$

$$= -\frac{1}{2}\int 2\cdot u^{13} - u^{14}\, du = -\frac{1}{2}\left(\frac{2\cdot u^{14}}{14} - \frac{u^{15}}{15}\right) + C$$

$$= -\frac{1}{14}(2-x^2)^{14} + \frac{1}{30}(2-x^2)^{15} + C$$

c) $\sin^2 x = \frac{1}{2}(1-\cos 2x)$ fordi $\cos(2x) = \cos^2 x - \sin^2 x$

$$\int \sin^2 x \, dx = \frac{1}{2}\int 1-\cos(2x)\, dx$$
$$= \frac{1}{2}\left(x - \frac{1}{2}\sin(2x)\right) + C$$
$$= \frac{x}{2} - \frac{1}{4}\sin(2x) + C$$

d) $\int x^3(2-x^2)^{13} dx$

$u = 2-x^2$, $x^2 = 2-u$
$u' = -2x$

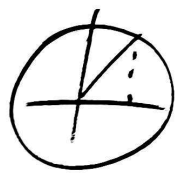
$$x^3 = \underbrace{x^2}_{2-u} \cdot \underbrace{x}_{-\frac{1}{2}u'}$$

$$= \int -\frac{1}{2}u'(2-u)u^{13}\, dx$$
$$= \int -\frac{1}{2}(2-u)u^{13}\, du$$
$$= -\frac{1}{2}\int 2\cdot u^{13} - u^{14}\, du = -\frac{1}{2}\left(\frac{2\cdot u^{14}}{14} - \frac{u^{15}}{15}\right) + C$$
$$= \frac{-1}{14}(2-x^2)^{14} + \frac{1}{30}(2-x^2)^{15} + C$$

$$\int_0^{\pi} \sin\left(\frac{x}{3}\right) dx + (-3)\int_0^{\pi} \cos(3x)\, dx$$

1 del $\quad \int_0^{\pi} \sin\left(\frac{x}{3}\right) dx \qquad u = \frac{x}{3} \qquad u' = \frac{1}{3}$

$$= \int_0^{\pi} \underset{u'}{\frac{1}{3}} \cdot 3\sin(u)\, dx = \int_0^{\pi/3} 3\sin(u)\, du$$



$$= -3\cos(u) \Big|_0^{\pi/3}$$

$$= -3\Big(\underset{1/2}{\cos(\pi/3)} - \underset{1}{\cos(0)}\Big) = -3\left(\frac{1}{2} - 1\right) = \frac{+3}{2}.$$

2 del : $\quad -\int_0^{\pi} \underset{v'}{3}\, \underset{\cos(v)}{\cos(3x)}\, dx \qquad v = 3x \qquad v' = 3$

$$= -\int_0^{3\pi} \cos(v)\, dv = -\sin v \Big|_0^{3\pi}$$

$$= -\left(\sin(3\pi) - \sin(0)\right) = 0$$

$$\int_0^{\pi} \sin\left(\frac{x}{3}\right) - 3\cos(3x)\, dx = \frac{3}{2} + 0 = \underline{\underline{\frac{3}{2}}}$$

$\text{Int} = \int \frac{x+5}{(x-3)(x+1)}\,dx$ skriver om integranden.

$\frac{x+5}{(x-3)(x+1)} = \frac{a}{x-3} + \frac{b}{x+1}$ (delbroks-oppspalting.)

ganger opp med $(x-3)(x+1)$

$x+5 = a(x+1) + b(x-3)$

$x+5 = (a+b)x + a - 3b$

I Likningssystem:

$a+b=1$

$a-3b=5$ etc...

II setter inn $x=-1$:

$-1+5 = a\cdot 0 + b(-1-3)$

$4 = b(-4)$

så $\underline{b=-1}$

$x=3$ $\quad 8 = a\cdot 4 + b\cdot 0$ så $\underline{a=2}$

$\int \frac{2}{x-3} - \frac{1}{x+1}\,dx = 2\ln|x-3| - \ln|x+1| + c$ ved lineære substitusjoner.

$$\int_0^1 \frac{x+5}{x^2-2x-3}\,dx = 2\ln|x-3| - \ln|x+1| \Big|_0^1$$

$$= 2\ln|-2| - \ln|2| - \big(2\ln|-3| - \underbrace{\ln(1)}_{0}\big)$$

$$= \ln(2) - 2\ln 3$$

$$= \ln 2 - \ln(3^2) = -(\ln 9 - \ln 2)$$

$$= \underline{-\ln\left(\tfrac{9}{2}\right)}$$