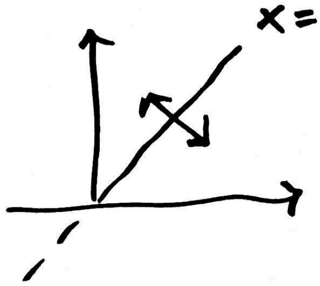
3 feb. 11E Derivasjon av sin, cos og tan.
26

vinkelenhet radianer.

1 $(\sin x)' = \cos x$

2 $(\cos x)' = -\sin x$

3 $(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = 1 + \tan^2 x = \frac{1}{\cos^2 x}$



$\cos x = \sin\left(\frac{\pi}{2} - x\right)$

Benytter kjerneregelen:

$$(\cos x)' = \left(\sin\left(\frac{\pi}{2} - x\right)\right)'$$

$$= \cos\left(\frac{\pi}{2} - x\right) \underbrace{\left(\frac{\pi}{2} - x\right)'}_{-1}$$

$$(\cos x)' = -\cos\left(\frac{\pi}{2} - x\right) = -\sin x$$

2 følger fra 1.

$$\tan x = \frac{\sin x}{\cos x} = \sin x \cdot \frac{1}{\cos x} = \sin x \cdot (\cos x)^{-1}$$

$$(\tan x)' = \underbrace{(\sin x)'}_{\cos x} \cdot \left(\frac{1}{\cos x}\right) + \sin x \cdot \underbrace{\left((\cos x)^{-1}\right)'}_{\frac{-1}{(\cos x)^2} \cdot (\cos x)'}$$

$$= \frac{\cos x}{\cos x} + \frac{(-1)^2 \sin^2 x}{\cos^2 x} = \frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x}$$

$$= \underline{1 + \tan^2 x} = \frac{1}{\cos^2 x} \quad \text{(ved Pytagoras)}$$

oppgaver

Deriver:

$f(x) = 2\sin x - 3\cos x + \sin(1)$

$f'(x) = 2(\sin x)' - 3(\cos x)' + \underbrace{(\sin 1)'}_{0}$

$= 2\cos x + 3\sin x$

$g(x) = 3\cos(2\pi x) - 25\sin\left(\frac{x}{5}\right)$

$g'(x) = 3(\cos(2\pi x))' - 25\left(\sin\frac{x}{5}\right)'$

$= 3(-\sin(2\pi x))\cdot\underbrace{(2\pi x)'}_{2\pi} - 25\cos\left(\frac{x}{5}\right)\cdot\underbrace{\left(\frac{x}{5}\right)'}_{\frac{1}{5}}$

$= \underline{-6\pi\sin(2\pi x) - 5\cos\left(\frac{x}{5}\right)}$

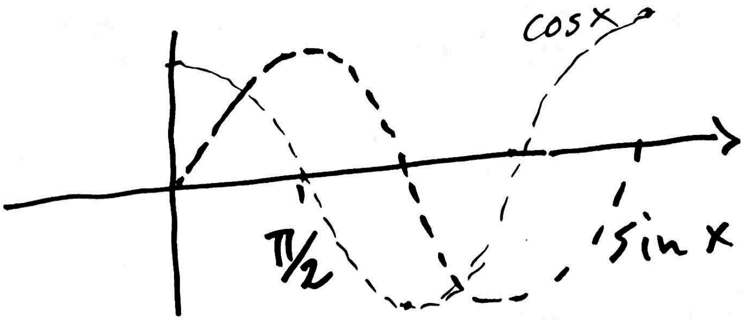
Hvis vinkelenheten er grader, da blir den deriverte av sin x:

$\sin\left(\frac{\pi}{180^\circ}v\right)$

$\left(\sin\left(\frac{\pi}{180^\circ}v\right)\right)' = \cos\left(\frac{\pi}{180^\circ}v\right)\cdot\left(\frac{\pi}{180^\circ}v\right)'$

$= \underline{\underline{\frac{\pi}{180^\circ}}}\cos\left(\frac{\pi}{180^\circ}v\right)$

For å slippe denne faktoren, bruker vi heller radianer.



sin x er voksende i $[0, \pi/2]$

cos x er avtagende i $[0, \pi/2]$.

$$(\sin x)' = +\cos x$$
$$(\cos x)' = -\sin x$$

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oppg. Deriver $\cos(x^2) + \cos^3(x)$.

$$(\cos x^2 + \cos^3 x)'$$
$$= (\cos(x^2))' + ((\cos x)^3)'$$
$$= -\sin(x^2)\cdot \underbrace{(x^2)'}_{2x} + 3(\cos x)^2 \underbrace{(\cos x)'}_{-\sin x}$$
$$= \underline{-2x\sin(x^2) - 3\cos^2 x \cdot \sin x}$$

oppg. Deriver $h(x) = x^2 \sin(2-x)$

$$h'(x) = (x^2)'\sin(2-x) + x^2(\sin(2-x))'$$
$$= 2x\sin(2-x) + x^2\cos(2-x)\underbrace{(2-x)'}_{-1}$$
$$= \underline{2x\sin(2-x) - x^2\cos(2-x)}$$

opg Deriver $j(x) = 2\cos^2 x - \cos(2x)$.

$$= 2(\cos x)^2 - \cos(2x)$$

$$j'(x) = 2 \cdot 2(\cos x) \cdot \underbrace{(\cos x)'}_{-\sin x} - (-\sin(2x)) \cdot \underbrace{(2x)'}_{2}$$

$$= \underline{-4\sin x \cos x + 2\underbrace{\sin(2x)}_{2\sin x \cdot \cos x}}$$

$$j'(x) = 0$$

($j(x)$ er en konstant funksjon :

$$2\cos^2 x - (\cos^2 x - \sin^2 x) = \cos^2 x + \sin^2 x$$
$$= 1 \text{ for alle } x$$

Derfor er $j'(x) = (1)' = 0$)

Bevis for at $(\sin x)' = \cos x$

$\longrightarrow$

$$(\sin x)' = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

addisjonsformel for sin gir:

$$\lim_{h\to 0} \frac{\sin(x)\cos(h) + \sin(h)\cos x \quad - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x\,(\cos(h) - 1)}{h} + \frac{\sin(h)\cos x}{h}$$

$$= \sin x \lim_{h\to 0} \frac{\cos(h) - 1}{h} + \cos x \lim \frac{\sin(h)}{h}.$$

Resultat $\lim_{h\to 0} \frac{\sin h}{h} = 1$

$$\lim_{h\to 0} \frac{1-\cos(h)}{h^2} = \frac{1}{2}$$

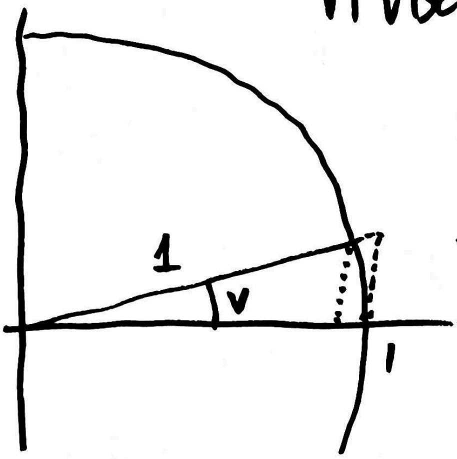
$$\left(\text{så} \quad \lim_{h\to 0} \frac{1-\cos(h)}{h} = \lim_{h\to 0} \frac{1-\cos h}{h^2}\cdot h \quad = 0\right)$$

Fra dette resultatet får vi

$$(\sin(x))' = \sin x \underbrace{\lim_{h\to 0} \frac{\cos(h)-1}{h}}_{0} + \cos x \underbrace{\lim_{h\to 0} \frac{\sin(h)}{h}}_{1}$$

$$= \underline{\cos x}$$

Vi viser nå grensene.



$\} \sin v$ $\qquad v > 0$

lille trekant < areal sirkelsegment < største trekant

$$\frac{1}{2}\sin v \cos v < \frac{1}{2} v < \frac{1}{2}\cdot 1 \cdot \underbrace{\tan v}_{\frac{\sin v}{\cos v}} \quad , \text{for } v > 0$$

deler med v, ganger med 2

$$\frac{\sin v}{v}\cos v < 1 < \frac{\sin v}{v}\cdot\frac{1}{\cos v}$$

$$\frac{\sin v}{v} < \frac{1}{\cos v} \qquad \cos v < \frac{\sin v}{v}$$

så $\quad \cos v < \dfrac{\sin v}{v} < \dfrac{1}{\cos v}$

siden $\lim\limits_{v \to 0} \cos v = 1$

og $\lim\limits_{v \to 0} \dfrac{1}{\cos v} = 1$

så må $\quad \lim\limits_{v \to 0^+} \dfrac{\sin v}{v} = 1$

$$\frac{\sin(-v)}{-v} = \frac{-\sin v}{-v} = \frac{\sin v}{v}, \text{ så } \lim_{v \to 0} \frac{\sin v}{v} = 1.$$

Vi viser nå at $\lim_{h\to 0} \frac{1-\cos(h)}{h^2} = \frac{1}{2}$

$$\frac{1-\cos(h)}{h^2} = \frac{(1-\cos(h))}{h^2}\frac{(1+\cos(h))}{(1+\cos(h))}$$

$$= \frac{1-\cos^2(h)}{h^2(1+\cos(h))} \overset{\text{Pytagoras}}{=} \frac{\sin^2(h)}{h^2(1+\cos(h))}$$

$$= \left(\frac{\sin(h)}{h}\right)^2 \cdot \frac{1}{1+\cos(h)} .$$

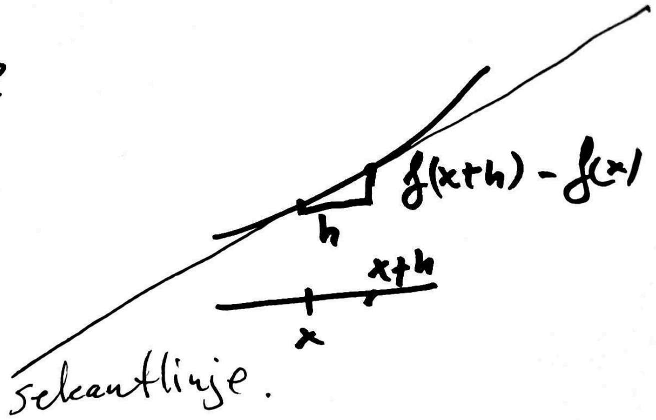
Derfor er $\lim_{h\to 0} \frac{1-\cos(h)}{h^2} = \lim_{h\to 0} \left(\frac{\sin(h)}{h}\right)^2 \frac{1}{1+\cos(h)}$

$$\overset{\text{grense-setningene}}{=} \left(\lim_{h\to 0} \frac{\sin h}{h}\right)^2 \cdot \lim_{h\to 0} \frac{1}{1+\cos(h)}$$

$$= 1^2 \cdot \frac{1}{1+1} = \frac{1}{2} .$$

Definisjon av den deriverte

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$



Derivasjon er lineær:

$$\left(a \cdot f(x) + b \cdot g(x)\right)' = a f'(x) + b \cdot g'(x)$$

Kjerneregelen: $\left(f(u(x))\right)' = (f \circ u(x))'$

$$= f'(u(x)) \cdot u'(x).$$

Produktregelen: $f(x) \cdot g(x) = (f \cdot g)(x)$

$$\left(f(x)\, g(x)\right)' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$