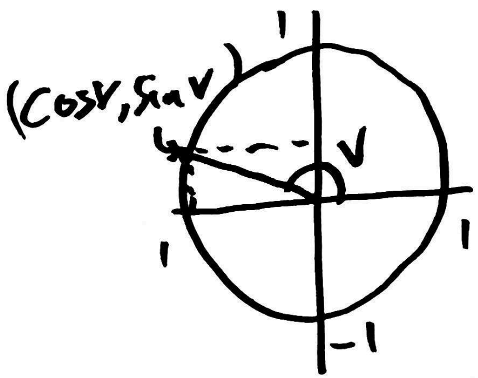
21 jan
26

10G Trigonometriske identiteter

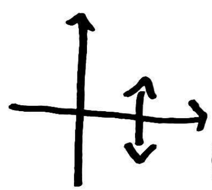


kvadranter

2	1
3	4

Pytagoras (enhetsformel) $\cos^2 v + \sin^2 v = 1$ for alle v

$((\cos v)^2 + (\sin v)^2 = 1)$



speglar om x-aksen

$(x, y) \leftrightarrow (x, -y)$

vinklen $v \leftrightarrow -v$

$\cos(-v) = \cos v$

$\sin(-v) = -\sin v$

Addisjonsformlene

$$\sin(x+y) = \sin x \cdot \cos y + \sin y \cdot \cos x$$
$$\sin(x-y) = \sin x \cdot \cos y - \sin y \cdot \cos x$$

kombinert: $\sin(x \pm y) = \sin x \cos y \pm \sin y \cos x$.

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

Denne følger fra addisjonsformelen for sinus.

$$\cos(x+y) = \sin(90^\circ - (x+y)) = \sin((90^\circ - x) - y)$$
$$= \underbrace{\sin(90^\circ - x)}_{\cos x} \cdot \underbrace{\cos(-y)}_{\cos(y)} + \underbrace{\sin(-y)}_{-\sin y} \cdot \underbrace{\cos(90^\circ - x)}_{\sin x}$$
$$= \cos x \cdot \cos y - \sin y \sin x$$

$$\tan(x+y) = \frac{\sin(x+y)}{\cos(x+y)} = \frac{(\sin x \cos y + \sin y \cos x)/(\cos x \cdot \cos y)}{(\cos x \cos y - \sin x \sin y)/(\cos x \cdot \cos y)}$$
$$= \frac{\tan x + \tan y}{1 - \tan x \cdot \tan y}$$

x, y og $x+y$ $\neq 90^\circ + 180^\circ \cdot n$

uttrykk $\sin(v+30^\circ)$ ved hjelp av $\sin v$ og $\cos v$

$$= \sin v \underbrace{\cos(30^\circ)}_{\sqrt{3}/2} + \underbrace{\sin(30^\circ)}_{1/2} \cos v$$

$$\sin(v+30^\circ) = \frac{\sqrt{3}}{2}\sin v + \frac{1}{2}\cos v$$

$$= \frac{1}{2}\left(\sqrt{3}\sin v + \cos v\right)$$

$$A \sin v + B \cos v = C \sin(v+k)$$

for passende C og k.

($B/A = \tan k \ldots$)

$A^2 + B^2 = C^2$

$$\sin 15^\circ = \cos(75^\circ) = \cos(30^\circ + 45^\circ)$$

$$= \cos(30^\circ)\cos(45^\circ) - \sin(30^\circ)\sin(45^\circ)$$

$$\sqrt{3}/2 \cdot 1/\sqrt{2} - 1/2 \cdot 1/\sqrt{2}$$

$$= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}} \cdot \left(\frac{\sqrt{2}}{\sqrt{2}}\right) = \underline{\frac{\sqrt{6}-\sqrt{2}}{4}} \quad \sim \underline{0.259}$$

$$\cos(15^\circ) = \sin(75^\circ) = \frac{\sqrt{6}+\sqrt{2}}{4} \sim \underline{0.966}$$

eks.

$$\tan(45^\circ + v) = \frac{\tan(45^\circ) + \tan(v)}{1 - \tan(45^\circ) \cdot \tan v}$$

$$= \frac{1 + \tan v}{1 - \tan v} \qquad \tan v \neq 1$$

$$\cos(v + 7^\circ) = \cos(7^\circ)\cos v - \sin(7^\circ)\sin v$$

$$= 0.9925 \cos v - 0.1219 \sin v.$$

$$\sin(180^\circ - v) = \underbrace{\cos(180^\circ)}_{(-1)}\underbrace{\sin(-v)}_{(-\sin v)} + \cos(-v)\underbrace{\sin(180^\circ)}_{0}$$

$$= \sin v \quad \checkmark$$

Refleksjon om
y-aksen

Dobling av vinkel formler

$\sin(2x) = \sin(x+x) = 2\sin x \cos x$

$\cos(2x) = \cos(x+x) = \cos^2 x - \sin^2 x$

Kombinert med Pytagoras $\cos^2 x + \sin^2 x = 1.$

$\left(\begin{array}{l} \sin^2 x = 1 - \cos^2 x \\ \cos^2 x = 1 - \sin^2 x \end{array} \right)$

$\cos(2x) = (1 - \sin^2 x) - \sin^2 x = 1 - 2\sin^2 x$

$2\sin^2 x = 1 - \cos(2x)$ $\quad \sin^2 x = \dfrac{1 - \cos(2x)}{2}$

Tilsvarende $\cos^2 x = \dfrac{1 + \cos(2x)}{2}.$

$\sin^2(15^\circ) = \dfrac{1 - \cos(\overbrace{2 \cdot 15^\circ}^{30^\circ})}{2} = \dfrac{1 - \sqrt{3}/2}{2} = \dfrac{2 - \sqrt{3}}{4}, \quad \sin(15^\circ) > 0$

$\sin(15^\circ) = \sqrt{\dfrac{2 - \sqrt{3}}{4}} = \dfrac{\sqrt{2 - \sqrt{3}}}{2}.$

tidligere: $\sin(15^\circ) = \frac{\sqrt{6}-\sqrt{2}}{4}$!

$$\sin^2(15^\circ) = \left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)^2 = \frac{6+2-2\sqrt{2}\sqrt{8}}{4^2}$$

$$= \frac{8-2\sqrt{2}\sqrt{2}\cdot\sqrt{3}}{4\cdot 4} = \frac{4(2-\sqrt{3})}{4\cdot 4} = \frac{2-\sqrt{3}}{4} \checkmark$$

så $\frac{\sqrt{6}-\sqrt{2}}{4} = \frac{\sqrt{2-\sqrt{3}}}{2}$

$$\sin^2(7.5^\circ) = \frac{1}{2}\Big(1-\cos \overbrace{15^\circ}^{2\cdot 7.5^\circ}\Big) = \frac{1}{2}\left(1 - \frac{\sqrt{6}+\sqrt{2}}{4}\right)$$

$\sin 7.5^\circ > 0$

$$\sin(7.5^\circ) = \sqrt{\frac{4-\sqrt{6}-\sqrt{2}}{8}} = \underline{\frac{1}{2}\sqrt{\frac{4-\sqrt{6}-\sqrt{2}}{2}}}$$

Eksempel

$$\cos(3x) = \cos(x+2x) = \cos x \cos 2x - \sin x \sin 2x$$

$$= \cos x(\cos^2 x - \sin^2 x) - \sin x(2\sin x \cos x)$$

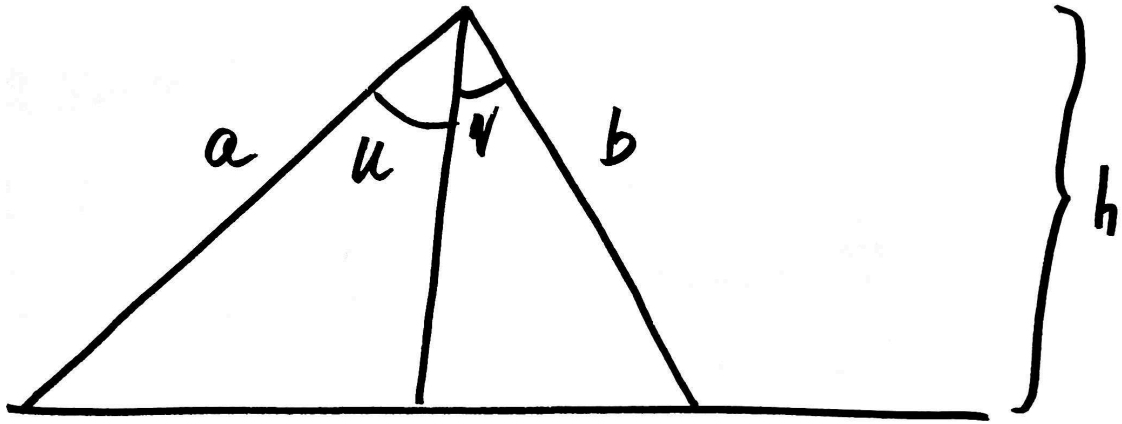
$$= \cos^3 x + \cos x(-\sin^2 x) - 2\sin^2 x \cos x$$

$$= \cos^3 x - 3\cos x(\underbrace{1-\cos^2 x}_{\sin x}).$$

$$\cos(3x) = \underline{4\cos^3 x - 3\cos x}$$

delvis bevis for addisjonsformelen for sinus

$h = b\cos v = a\cos u$



Totalt areal til den store trekanten

$\frac{1}{2}ab\sin(u+v)$ = summen av arealet til de to mindre trekantene.

$= \frac{1}{2}\underset{\underset{a\cos u}{\uparrow}}{h}\cdot b\sin v + \frac{1}{2}\underset{\underset{b\cos v}{\uparrow}}{h}\,a\sin u$

deler ut med $\frac{1}{2}ab$

$\sin(u+v) = \cos u \cdot \sin v + \cos v \sin u$

addisjonsformelen for

$0 \leq u, v \leq 90°$

utvides til alle u, v ...

oppg Finn vinkler v slik at $\sin x = \sin 2x$.

$\sin x = 2\sin x \cos x$

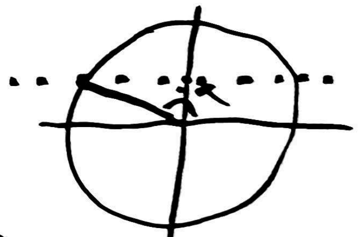
$\sin x\,(1 - 2\cos x) = 0$

$\Leftrightarrow \sin x = 0$ eller $\cos x = \frac{1}{2}$

$x = 0^\circ + 360^\circ \cdot n$ $\quad$ $x = 60^\circ + 360^\circ \cdot n$

$x = 180^\circ + 360^\circ \cdot n$ $\quad$ $x = -60^\circ + 360^\circ \cdot n$.

eks. Finn eksakt verdi til $\sin 2x$ når $\sin x = \frac{2}{5}$ og x ligger i 2. kvadrant.



Pythagoras

$\cos^2 x + \sin^2 x = 1$

$\cos x < 0$ siden x er i 2. kvadrant

$\cos^2 x = 1 - \sin^2 x = 1 - \frac{4}{25} = \frac{21}{25}$

$\cos x = -\sqrt{\frac{21}{25}} = -\frac{\sqrt{21}}{5}$

$\sin 2x = 2\sin x \cos x$

$= 2 \cdot \frac{2}{5} \cdot \left(-\frac{\sqrt{21}}{5}\right)$

$= \frac{-4\sqrt{21}}{25}$

Øving

Oppg 10.108

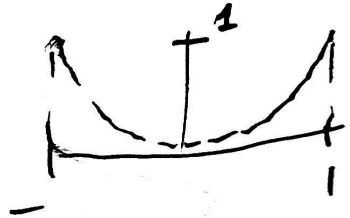
$$f(v) = 1 + 5\cos^2 v + 3\sin^2 v$$
$$= 1 + 2\cos^2 v + 3\underbrace{(\cos^2 v + \sin^2 v)}_{1}$$
$$= \underline{4 + 2\cos^2 v} \quad (= 6 - 2\sin^2 v)$$

Pytagoras:
$\cos^2 v + \sin^2 v = 1$
for alle v

a) $0 \leq \cos^2 v \leq 1$ for alle v

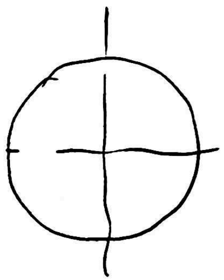
$v = 0°$: $\cos^2 0° = 1$

$v = 90°$: $\cos^2(90°) = 0$



minste verdi til $f(v)$ er 4

største —— er 6



b) Hvor realiseres største og minste verdi $[0, 360°]$?

minste verdi : $\cos^2 v = 0 \Leftrightarrow \cos v = 0$: $v = 90°$ og $270°$

største verdi : $\cos^2 v = 1 \Leftrightarrow \cos v = \pm 1$: $v = 0°, 180°$
$v = 360°$.