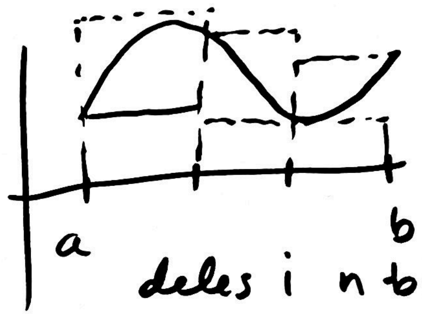
2 april 25

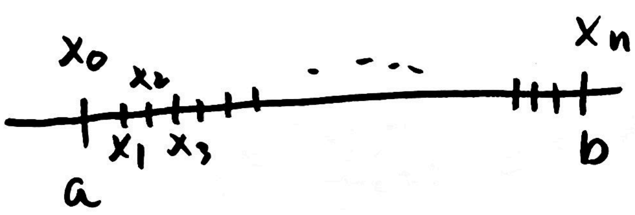
15B Numerisk integrasjon

Bestemt integral $\int_a^b f(x)\,dx$



deles i n biter

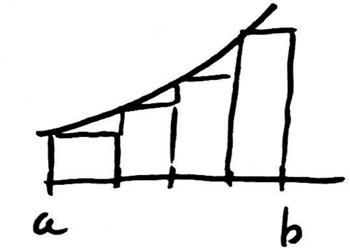
$n = 3$



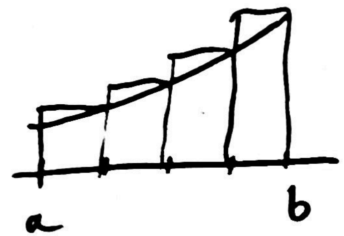
bredde til hvert delintervall

$\Delta x = \frac{b-a}{n}$ $\qquad x_i = a + \Delta x \cdot i$

Venstre rektangelmetode $\quad V_n = \sum_{i=0}^{n-1} f(x_i) \cdot \Delta x$



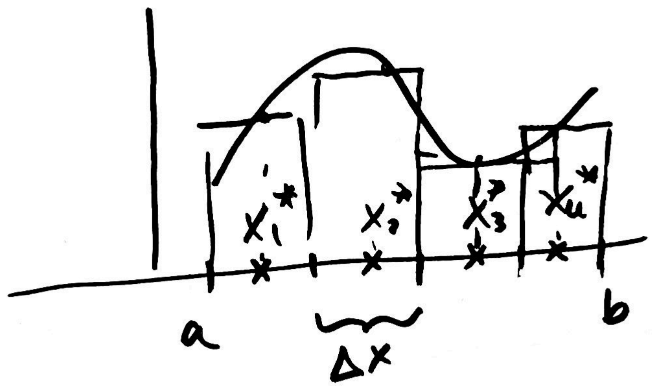
Høyre —— 〃 —— $\quad H_n = \sum_{i=1}^{n} f(x_i)\,\Delta x$



Gir nøyaktig svar for konstante funksjoner.

$n \geq 1$

Midtpunktmetoden
(rektangelmetode)

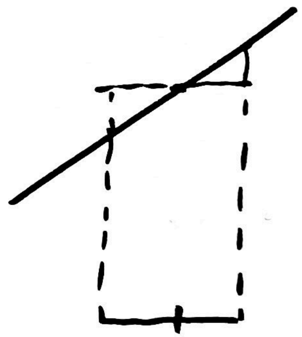


$n = 4$

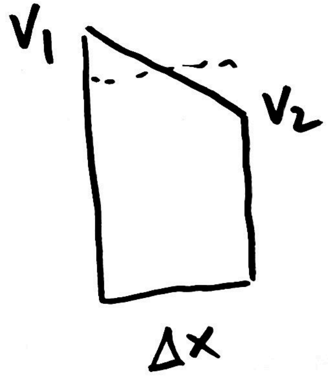
$$x_i^* = a + \frac{\Delta x}{2} + \Delta x(i-1)$$
$$= a + \Delta x \cdot i - \frac{\Delta x}{2}$$

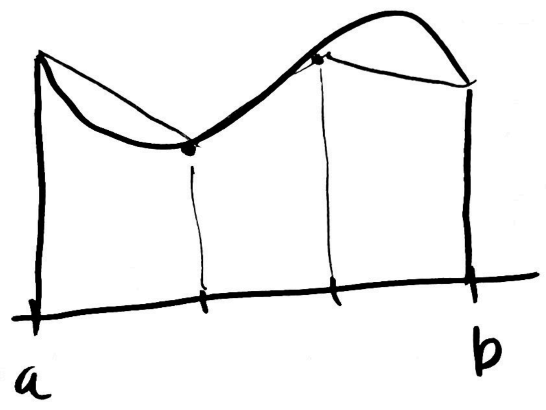
$$M_n = \sum_{i=1}^{n} f(x_i^*) \cdot \Delta x$$

Eksakt på lineære funksjoner



Tropesmetoden





$n = 3$

areal med fortegn til trapeset er lik $\frac{V_1 + V_2}{2} \cdot \Delta x$

$$T_n = \sum_{i=0}^{n-1} \frac{(f(x_i) + f(x_{i+1}))}{2} \cdot \Delta x$$

$$= \frac{1}{2}\left(\sum_{i=0}^{n-1} f(x_i) \cdot \Delta x + \underbrace{\sum_{i=0}^{n-1} f(x_{i+1}) \Delta x}_{\sum_{j=1}^{n} f(x_j)} \right)$$

$$T_n = \frac{1}{2}(V_n + H_n)$$

$$H_n = V_n - f(x_0)\Delta x + f(x_n) \cdot \Delta x$$

$$= V_n + (f(b) - f(a))\Delta x$$

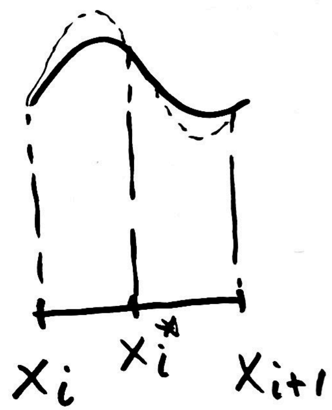
setter inn og får : $\underline{T_n = V_n + \frac{f(b) - f(a)}{2} \Delta x}$

Feil ved trapesmetoden

$$\frac{M_2 (b-a)^3}{12 n^2} = \frac{M_2}{12}(b-a)(\Delta x)^2$$

hvor $M_2 = \max_{x \in [a,b]} |f''(x)|$

Simpsons metode



Ønsker å vekte $f(x_i)$, $f(x_{i+1})$

og $f(x_{i+1}^*)$

slik at vi får

eksakt estimat

av integralet på 2.grads uttrykk!

$$\frac{c\,(f(x_i) + f(x_{i+1})) + d\, f(x_{i+1}^*)}{2c+d}$$

er eksakt på lineære uttrykk.

Vektingen er $\frac{1}{6}$ $\frac{4}{6}$ $\frac{1}{6}$

$$\frac{1}{6}\left(f(x_i) + 4f(x_{i+1}^*) + f(x_{i+1})\right)$$

$$S_n = \frac{\Delta x}{6}\Big(f(a) + 4f\left(a+\frac{\Delta x}{2}\right) + f(a+\Delta x)$$
$$f(a+\Delta x) + 4f\left(a+\Delta x+\frac{\Delta x}{2}\right) + f(a+2\Delta x)$$
$$\dots$$

vekting: $\frac{1}{6}(1, 4, 2, 4, 2, 4, 2, \dots\ 2, 4, 1)$

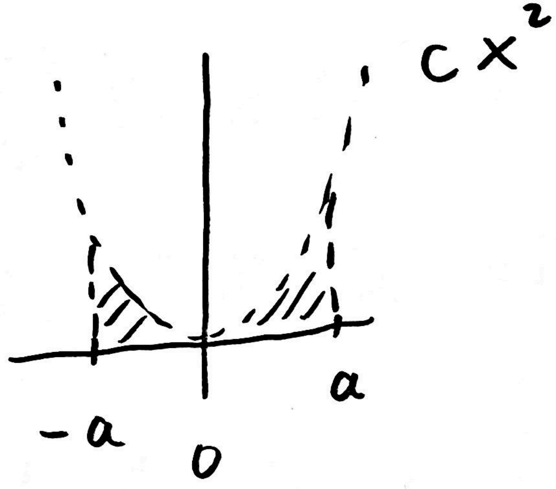
$$= \frac{1}{6}(2 \cdot T_n + 4M_n) = \frac{1}{3}T_n + \frac{2}{3}M_n$$

$$S_n = \frac{1}{3}T_n + \frac{2}{3}M_n$$

$M_4 = \max\limits_{x\in[a,b]} |f^{(4)}(x)|$

S_n simpsons estimat gir eksakt resultat for alle (allen) polynomer av grad ≤ 3.

Feilen ved simpsons metode $\leq \underline{\dfrac{M_4 (b-a)^5}{180\, n^4}} = \dfrac{M_4}{180}(\Delta x)^4 (b-a)$



$$\int_{-a}^{a} cx^2\,dx$$

$$= c\cdot 2\int_0^a x^2\,dx = c\,2\,\frac{a^3}{3}$$

$$= \underline{\frac{2c}{3}a^3}$$

Ved simpsons metode :

$$\frac{f(-a) + 4f(0) + f(a)}{6}\cdot \Delta x$$

$$\frac{c(-a)^2 + 0 + ca^2}{6}\cdot 2a$$

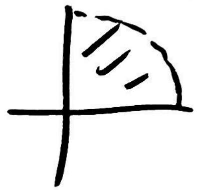
$$= \frac{2c\,a^2}{6}\cdot 2a$$

$$= \frac{2c}{3}a^3 \quad ✓$$

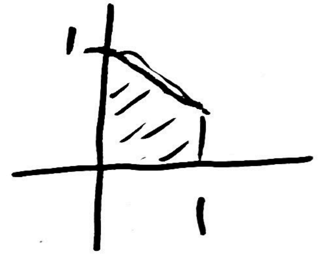
eksakt verdi.

$$\int_0^{\pi} \sin x \, dx = 2$$

$$4\int_0^1 \sqrt{1-x^2} \, dx = \pi$$

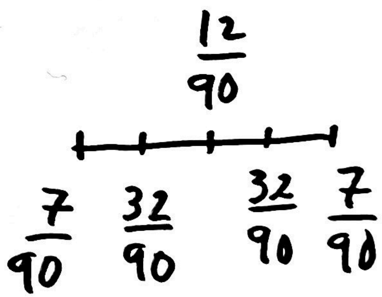


$$4\int_0^1 \frac{1}{1+x^2} \, dx = \pi$$



$$\left(1+\int_0^1 e^x \, dx = e\right)$$

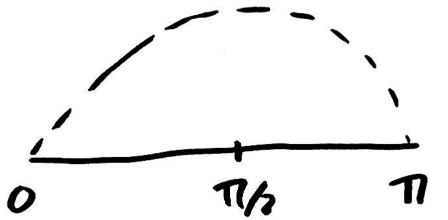
Til orientering:



En slik vekting
i hvert delintervall
gir eksakt integral for
polynomer av grad ≤5

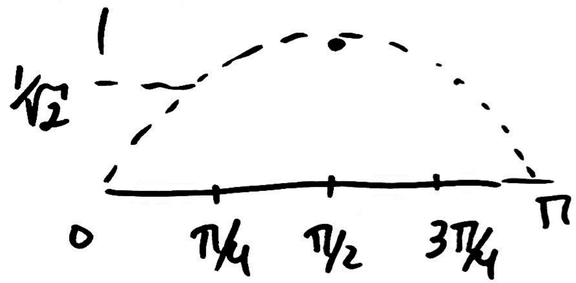
Regner med Simpsons metode "for hånd"

$n=1$ $\quad \int_0^{\pi} \sin x \, dx$



$$S_1 = \pi \left(\frac{1 \cdot \sin 0 + 4 \cdot \sin \pi + 1 \cdot \sin \pi}{6} \right) = \frac{2\pi}{3} \sim \underline{2.094}$$

$n=2$



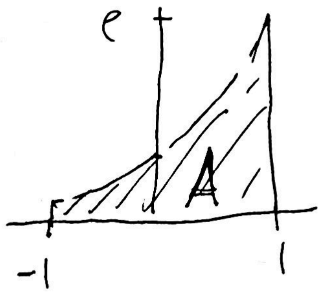
$$S_2 = \underset{\substack{\uparrow \\ \Delta x}}{\frac{\pi}{2}} \frac{1}{6} \left(\sin 0 + 4 \cdot \sin \frac{\pi}{4} + 2 \sin \frac{\pi}{2} + 4 \sin \frac{3\pi}{4} + \sin \pi \right)$$

$$= \frac{\pi}{12} \left(2 + 2 \cdot 4 \cdot \frac{1}{\sqrt{2}} \right) = \frac{\pi}{12} \left(2 + 4\sqrt{2} \right)$$

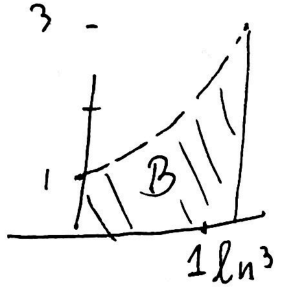
$$= \frac{\pi}{6} \left(1 + 2\sqrt{2} \right)$$

$$\sim \underline{2.0045\ldots}$$

15.9 a) $A = \int_{-1}^{1} e^x \, dx$



b)



$B = \int_0^{\ln 3} e^x \, dx$

c) $\int_1^2 e^x \, dx \sim 4.67077\ldots$