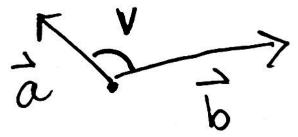
10 mars 25

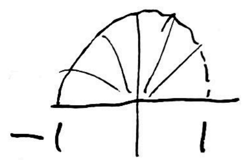
13D Avstand mellom punkt, linjer og plan.

Vinkel mellom vektorer og mellom linjer.



$0^\circ \leq v \leq 180^\circ$ "korteste vinkel mellom vektorene"

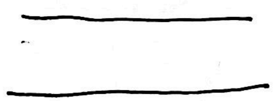
$\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cos(v)$



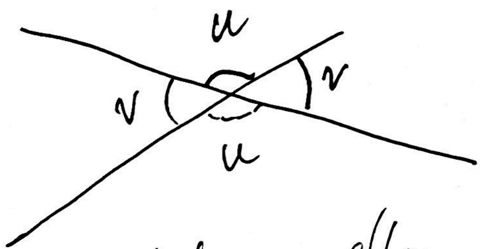
$\vec{a}, \vec{b} \neq \vec{0}$.

$$\underline{v = \arccos\left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}\right)}$$

parallelle



vinkelen mellom dem er 0



$u + v = 180^\circ$

$u, v \geq 0^\circ$

Vinkelen mellom to (ikke-parallelle) linjer er den minste av vinklene u og v

Vinkelen er mellom $\underline{0^\circ \text{ og } 90^\circ}$

eks To linjer i rommet

$$l_1 \begin{cases} x = 2t + 1 \\ y = -3t + 2 \\ z = 4t + 3 \end{cases} \qquad t \in \mathbb{R}$$

$$l_2 = [1, 2, -2]s + [1, 2, 3] \qquad s \in \mathbb{R}$$

De har retningsvektorer $\vec{r_1} = [2, -3, 4]$ og $\vec{r_2} = [1, 2, -2]$.

Vinkelen mellom retningsvektorene v:

$$\vec{r_1} \cdot \vec{r_2} = |\vec{r_1}||\vec{r_2}| \cos v.$$

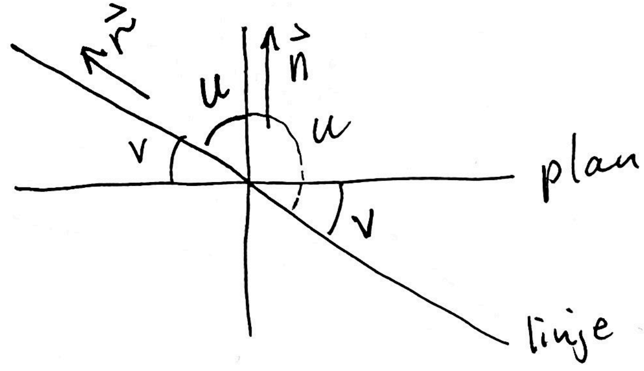
$$2 - 6 - 8 = \sqrt{2^2 + (-3)^2 + 4^2}\sqrt{1^2 + 2^2 + 2^2} \cos v$$

$$-12 = \sqrt{29 \cdot 9} \cos v$$

$$v = \arccos\left(\frac{-12}{\sqrt{261}}\right) = \arccos(-0.74278\ldots)$$

$$= 137.968^\circ$$

Vinkelen mellom linjene med retningsvektorer $\vec{r_1}$ og $\vec{r_2}$ er derfor $180^\circ - v = 42.031^\circ$



$v = 90° - u \qquad 0 \le u \le 90°$

$v = u - 90° \qquad 90° \le u \le 180°$

$$\underline{v = |90° - u|}$$

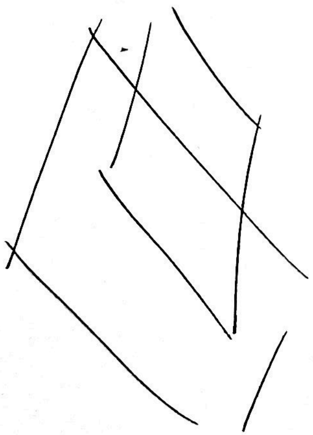
Vinkelen u mellom retningsvektoren $\vec{r} = [2,2,1]$ til linje
og normalvektoren $\vec{n} = [2,-3,0]$ - planet
er:

$$|\vec{r}||\vec{n}|\cos u = \vec{r}\cdot\vec{n} = 2\cdot 2 + 2(-3) = -2$$

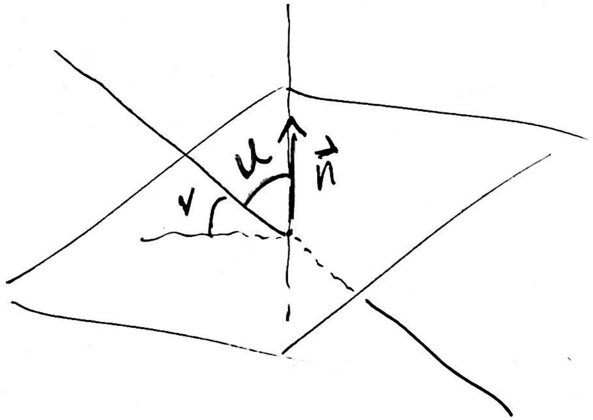
$$\cos u = \frac{-2}{\sqrt{9}\sqrt{13}} = \frac{-2}{3\sqrt{13}}$$

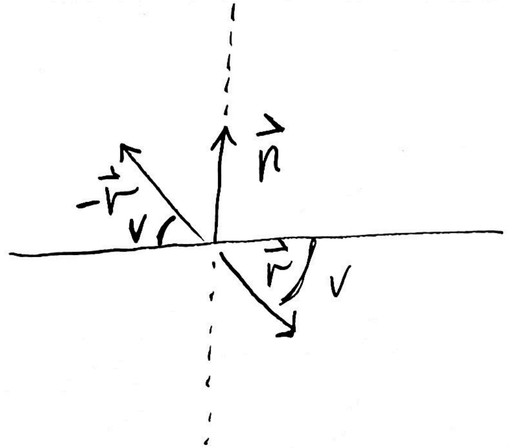
$$u = \arccos\left(\frac{-2}{3\sqrt{13}}\right) = \underline{100.655°}$$

$$v = |90° - u| = u - 90° = \underline{10.655°}$$



Vinkelen mellom to plan er
vinkelen mellom linjene som er parallelle til normalvektorene
til plana.

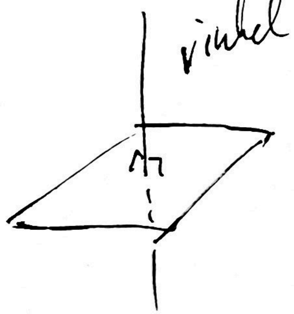




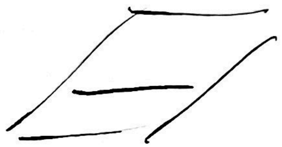
normalvektor til planet

$$\left| 90^\circ - \text{vinkelen mellom } \vec{n} \text{ og } \vec{r} \right|$$

retningsvektor til linjen



vinkel 90°



vinkel 0°.

oppgave

Finn vinklen mellom plant

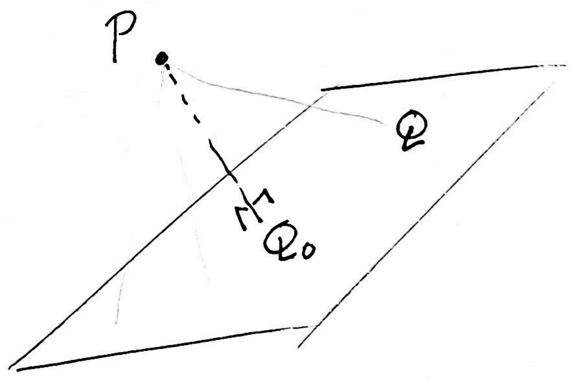
$$2x - 3y = 5$$

og linjen $[x, y, z] = [2, 2, 1]t + \vec{0}$

$\vec{r}$ retningsvektor for linjen.

$$2x - 3y = [2, -3, 0] \cdot [x, y, z] = 5 = [2, -3, 0] \cdot [1, -1, 0].$$

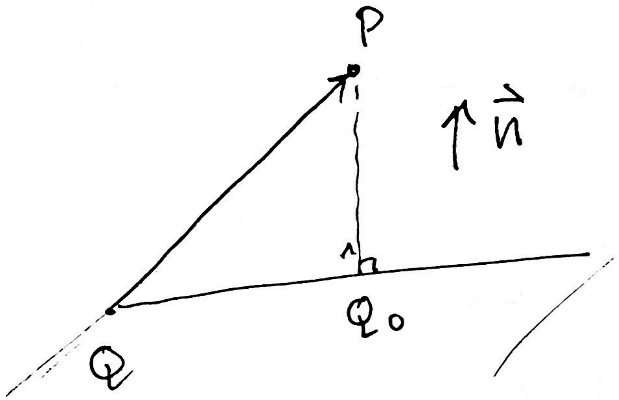
normalvektor til planet.

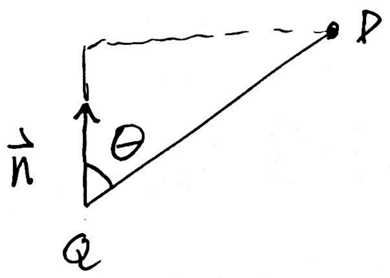


plan.

$$\min_{Q \text{ i planet}} |PQ| = \text{avstanden mellom } P \text{ og planet.}$$

Dette er avstanden $|PQ_0|$ hvor $\overrightarrow{Q_0P}$ står vinkelrett på planet.





$\overrightarrow{Q_0P}$ er komponenten til $\overrightarrow{QP}$ langs normalvektor $\vec{n}$.

$$|\overrightarrow{Q_0P}| = |\overrightarrow{QP}|\,|\cos\theta|$$

$$\cos\theta = \left| \frac{\vec{n} \cdot \overrightarrow{QP}}{|\vec{n}|\,|\overrightarrow{QP}|} \right| = \frac{|\vec{n} \cdot \overrightarrow{QP}|}{|\vec{n}|\,|\overrightarrow{QP}|}$$

$$|\overrightarrow{Q_0P}| = \frac{|\vec{n} \cdot \overrightarrow{QP}|}{|\vec{n}|}$$

Eksempel plan $2x - y + 2z = 6$

punktet $Q(1,2,3)$ er i planet

det er også $(3,0,0)$ ...

Finn avstanden mellom planet og punktet $P(1,1,1)$

$$\vec{QP} = \vec{OP} - \vec{OQ} = [1,1,1] - [1,2,3] = [0,-1,-2]$$
$$= -[0,1,2].$$

normalvektor til planet $\vec{n} = [2,-1,2]$

Avstanden er
$$\frac{|\vec{n} \cdot \vec{QP}|}{|\vec{n}|} = \frac{|-(0-1+4)|}{\sqrt{4+1+4}}$$
$$= \frac{|-3|}{\sqrt{9}} = \underline{\underline{\frac{3}{3}}} = \underline{\underline{1}}$$

Alternativt: Parametriserer linjen gjennom P med retningsvektor $\vec{n}$

$$\underbrace{[x,y,z]}_{\vec{OQ_0}} = \overbrace{[2,-1,2]}^{\vec{n}}t + \underbrace{[1,1,1]}_{\vec{OP}}$$

setter inn i likningen for planet og undersøker hvor vi treffer planet

$$2(2t+1) - (-t+1) + 2(2t+1) = 6$$
$$4t+2 + t - 1 + 4t + 2 = 6$$
$$9t = 6 - 3 = 3$$
$$t = \frac{3}{9} = \frac{1}{3}.$$

$$\vec{QP} = \vec{OP} - \vec{QQ_0} = -\vec{n}\,t = -\frac{1}{3}[2,-1,2]$$
$$|\vec{Q_0P}| = \frac{1}{3}|[2,-1,2]| = \frac{3}{3} = \underline{\underline{1}}$$

Avstanden mellom $ax + by + cz = d$ og origo er lik $\frac{|d|}{|[a,b,c]|}$

$$= \frac{|d|}{\sqrt{a^2+b^2+c^2}}$$

Alternativt

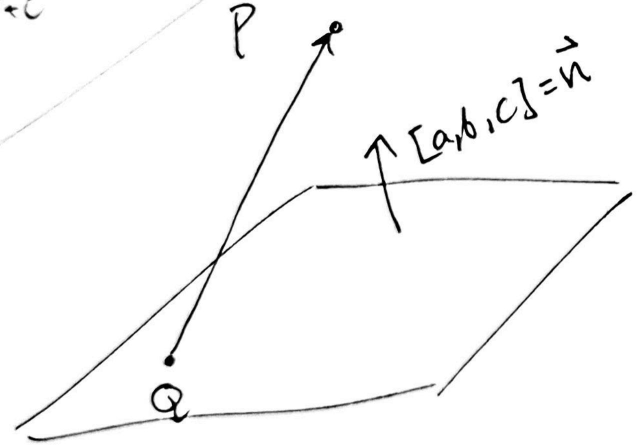
$ax+by+cz+d=0$

Avstanden mellom planet og (x,y,z) er

$$\frac{|ax+by+cz+d|}{\sqrt{a^2+b^2+c^2}}$$

Avstanden mellom planet gitt ved $ax+by+cz=d$ og punktet $P_0(x_0,y_0,z_0)$ er

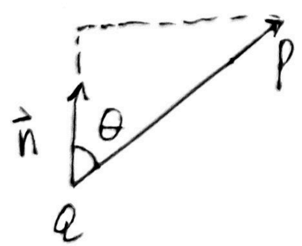
$$\frac{|ax_0+by_0+cz_0-d|}{\sqrt{a^2+b^2+c^2}}$$



Q i planet: $\vec{OQ}\cdot\vec{n}=d$

$$\frac{|\vec{QP}\cdot\vec{n}|}{|\vec{n}|}=\frac{|(\vec{OP}-\vec{OQ})\cdot\vec{n}|}{|\vec{n}|}$$

Dette gir formelen ovenfor.



$$\frac{\vec{n}\cdot\vec{QP}}{|\vec{n}|}=\frac{|\vec{n}||QP|\cdot\cos\theta}{|\vec{n}|}=|\vec{QP}|\cos\theta$$

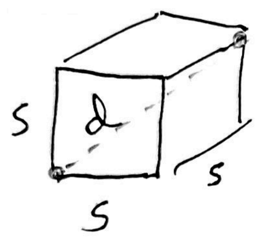
ta absolutt verdien.

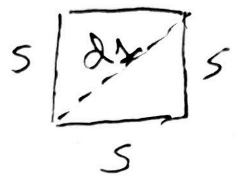
obligs

Ta gjerne med figurer, enhetssirkler ...

$d = 2\sqrt{3}$.

1. e)

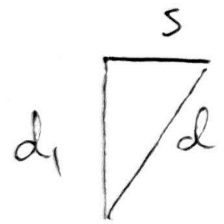




Pytagoras gir

$d_1^2 = \sqrt{s^2 + s^2} = \sqrt{2s^2}$

$d_1 = \sqrt{2}\, s$

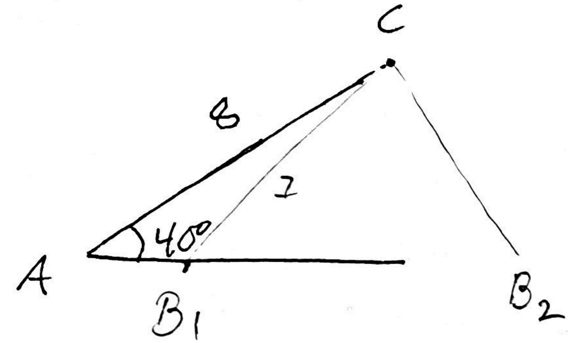


$d = \sqrt{s^2 + d_1^2} = \sqrt{s^2 + 2s^2} = \sqrt{3s^2} = \underline{\sqrt{3}\, s}$

$s = 2$

$\underline{V = 8}$ $\quad A = 4 \cdot 6 = \underline{24}$

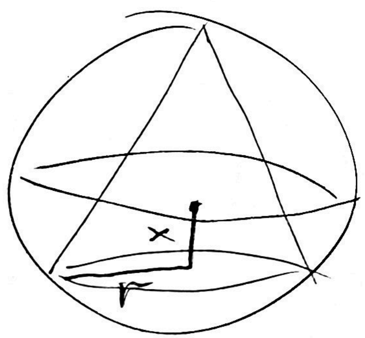
2e)



Typisk er bare B_2 funnet
B_1 er glemt.

4c) minst én vinkel som er 30° (ikke bare ~~n~~ødvendigvis én vinkel).

5



$r^2 + x^2 = R^2$

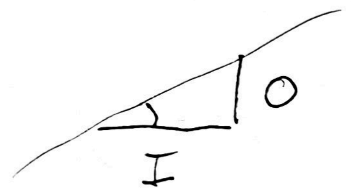
Areal grunnflate $\quad \pi \cancel{r}^2 = \pi(R^2 - x^2)$

høyden $\quad h = R + x$

Volum $\quad V = \underline{\frac{\pi}{3}(R^2 - x^2)(R + x)}$

$V'(x) = 0 \ldots$

6



$$\tan\theta = \frac{O}{I}$$

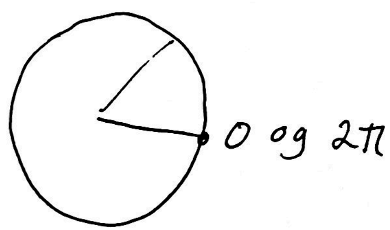
$$O = \tan\theta \, I$$

$$2O + I = 62\text{ cm}$$

$$\underbrace{2I\tan\theta + I}_{I(1+2\tan\theta)} = 62\text{ cm}$$

$$I = \frac{62\text{ cm}}{1+2\tan\theta}$$

7



$[0, 2\pi]$

9 a) $\sin^2 V = \frac{1}{2}$ gir $\sin V = \frac{1}{\sqrt{2}}$

og $\sin V = \frac{-1}{\sqrt{2}}$ (ofte glemt)

løs begge og kombiner løsningene (union)

$x^2 = (4 =) a$ likning

$|x| = \sqrt{x^2} = \sqrt{a}$ så $x = \sqrt{a}$

eller $x = -\sqrt{a}$

Løsningene er $\underline{\sqrt{a} \text{ og } -\sqrt{+a}}$.

10 $\arcsin x = 0.3786$ vinkel! enheten radianer (det står ikke $0.3786°$)

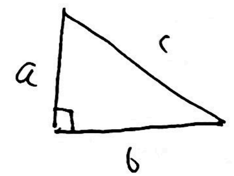
$x = \sin(0.3786)$

radianer.

11 a) $\sqrt{3}\sin x + \cos x > 0$

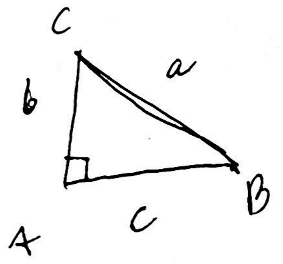
$\cos x(\sqrt{3}\tan x + 1) > 0$ (samt undersøk $\cos x = 0$!)

Det var noe unødig bruk av cos- og sinsetning.



cos setning → Pytagoras

$$a^2+b^2 \underbrace{-2ab\cos(90^\circ)}_{0} = c^2$$



$$\frac{a}{\sin A} = \frac{a}{\sin 90^\circ} = a = \frac{b}{\sin B} = \frac{c}{\sin C}$$

sier bare at

$b = a\sin B$ og $c = a\sin C$

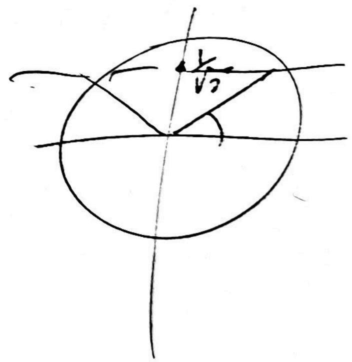
definisjon av sinus.

7 etc

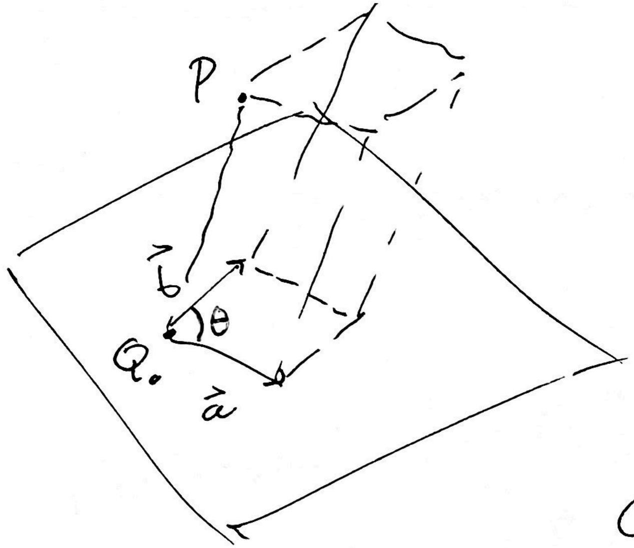
Føring av oppg. med trig. likninger

$\sin v = \frac{1}{\sqrt{2}}$ ~~$\arcsin\left(\frac{1}{\sqrt{2}}\right) = 45^\circ$~~

$$v_1 = \arcsin\left(\frac{1}{\sqrt{2}}\right) + 360^\circ \cdot n = \underline{45^\circ + 360^\circ \cdot n}$$



$$v_2 = 180^\circ - \arcsin\left(\frac{1}{\sqrt{2}}\right) + 360^\circ \cdot n$$
$$= \underline{135^\circ + 360^\circ \cdot n}$$



plan

parametrisering

$$\overrightarrow{OQ} = s\vec{a} + t\vec{b} + \overrightarrow{OQ_0} \qquad s, t \in \mathbb{R}$$

Avstanden fra P til planet er høyden i parallellepipedet utspent av $\vec{a}$, $\vec{b}$ og $\overrightarrow{Q_0P}$ med grunnflate parallellogram utspent av $\vec{a}$ og $\vec{b}$.

$$\text{høyden } h = \frac{\text{volum parallellepiped}}{\text{areal parallellogram}} = \frac{\left| \overrightarrow{Q_0P} \cdot (\vec{a} \times \vec{b}) \right|}{|\vec{a} \times \vec{b}|}$$

$$= \frac{\overrightarrow{Q_0P} \cdot \vec{n}}{|\vec{n}|} \qquad (\vec{a} \times \vec{b} = \vec{n} \text{ er en normalvektor til planet})$$

Oppg. Et punkt $P(-2,3,5)$

Plan parametrisert som $\begin{cases} x = s+t+1 \\ y = -2s+3t+4 \\ z = 4s-t-2 \end{cases}$ $s,t \in \mathbb{R}$

Finn avstanden fra P til planet.

$\vec{a} = [1,-2,4]$ og $\vec{b} = [1,3,-1]$ utspenner planet

En normalvektor $\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 4 \\ 1 & 3 & -1 \end{vmatrix}$

$$\cancel{\vec{n}} = \left[\begin{vmatrix} -2 & 4 \\ 3 & -1 \end{vmatrix}, -\begin{vmatrix} 1 & 4 \\ 1 & -1 \end{vmatrix}, \begin{vmatrix} 1 & -2 \\ 1 & 3 \end{vmatrix}\right]$$

$$= [2-12, -(-1-4), 3-(-2)]$$

$$= [-10, 5, 5] = 5[-2,1,1]$$

Benytter $\boxed{\vec{n} = [-2,1,1]}$

$Q(1,4,-2)$ ligger i planet $s=t=0$.

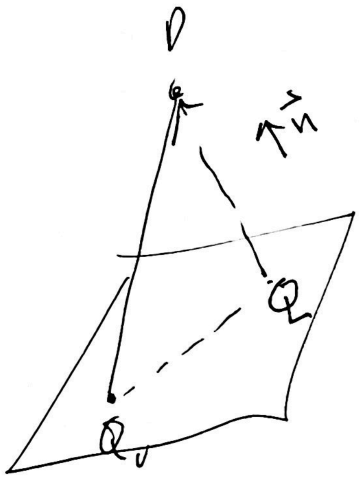
$\vec{QP} = \vec{OP} - \vec{OQ}$

$= [-2,3,5] - [1,4,-2]$

$\vec{QP} = [-3,-1,7]$

Avstanden fra P til planet: $\dfrac{|\vec{QP}\cdot[-2,1,1]|}{|[-2,1,1]|} = \dfrac{|6-1+7|}{\sqrt{4+1+1}} = \dfrac{12}{\sqrt{6}} = \underline{\underline{2\sqrt{6}}}$

Finn punktet Q på planet nærmest P.



$\overrightarrow{QP}$ = komponenten til $\overrightarrow{Q_0P}$ langs $\vec{n}$

$$\overrightarrow{OP} - \overrightarrow{OQ} = \frac{\overrightarrow{Q_0P} \cdot \vec{n}}{|\vec{n}|^2}\vec{n}$$

$$\text{så } \overrightarrow{OQ} = \overrightarrow{OP} - \frac{\overrightarrow{Q_0P} \cdot n}{|\vec{n}|^2}\vec{n}.$$

$$= [-2,3,5] - \underbrace{\frac{12}{6}}_{2}[-2,1,1]$$

$$= [-2,3,5] - [-4,2,2]$$

$$\overrightarrow{OQ} = \underline{[2,1,3]} \qquad \underline{Q(2,1,3)}$$

(Likning for planet $\quad -2x + y + z = -2(1) + 4 + (-2) = 0$ $\quad$ planet går gjennom origo.
(Q ligger i planet))