25 feb 25 — 12G

Vektorproduktet

kalles også kryssproduktet.

To 3-vektorer $\vec{a}$ og $\vec{b}$

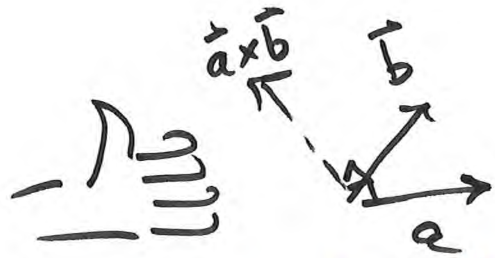
Vektorproduktet $\vec{a} \times \vec{b}$ er en 3-vektor som er ortogonal til $\vec{a}$ og $\vec{b}$

* $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$

θ vinkelen mellom $\vec{a}$ og $\vec{b}$

* $\vec{a}, \vec{b}, \vec{a} \times \vec{b}$ er et høyrehandsystem

Egenskaper: $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$



Ege $\vec{a} \times \vec{b}$ er lineær i både $\vec{a}$ og $\vec{b}$

$$(k\vec{a}) \times \vec{b} = k(\vec{a} \times \vec{b})$$

$$(\vec{a}_1 + \vec{a}_2) \times \vec{b} = \vec{a}_1 \times \vec{b} + \vec{a}_2 \times \vec{b}.$$

tilsvarende for $\vec{b}$.

$$\vec{a} \times \vec{a} = \vec{0}$$

Eksempel hvor vi benytter linearitet

$$(3\vec{a} - 2\vec{b}) \times (7\vec{a} + 4\vec{b})$$

$$= 3(\vec{a} \times (7\vec{a} + 4\vec{b})) - 2(b \times (7\vec{a} + 4\vec{b}))$$

$$= 3\cdot 7\,\vec{a}\times\vec{a} + 3\cdot 4\,\vec{a}\times\vec{b}$$
$$- 2\cdot 7\,\vec{b}\times\vec{a} - 2\cdot 4\,\vec{b}\times\vec{b}$$
$$= \vec{0} + 12\,\vec{a}\times\vec{b} - 14\underbrace{\vec{b}\times\vec{a}}_{-a\times b} + \vec{0}$$
$$= 12\vec{a}\times\vec{b} - 14(-\vec{a}\times\vec{b})$$
$$= \underline{26\vec{a}\times\vec{b}}$$

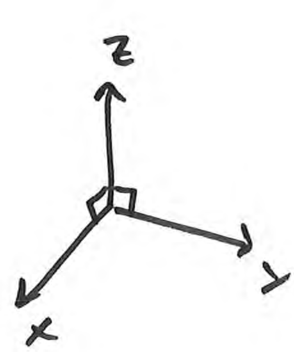
I går :

$$[a, b, 0]\times[c, d, 0]$$
$$= \left[0, 0, \begin{vmatrix} a & b \\ c & d \end{vmatrix}\right]$$

$\vec{e}_1 = [1, 0, 0]$, $\vec{e}_2 = [0, 1, 0]$, $\vec{e}_3 = [0, 0, 1]$

basisvektorer i x, y og z retning
(lengde 1)

$$[x, y, z] = x\vec{e}_1 + y\vec{e}_2 + z\vec{e}_3$$



$$\vec{e}_1\times\vec{e}_2 = \vec{e}_3$$
$$\vec{e}_1\times\vec{e}_3 = -\vec{e}_2$$
$$\vec{e}_2\times\vec{e}_3 = \vec{e}_1$$

$$[x_1, y_1, z_1] \times [x_2, y_2, z_2]$$

$$(x_1\vec{e}_1 + y_1\vec{e}_2 + z_1\vec{e}_3) \times (x_2\vec{e}_1 + y_2\vec{e}_2 + z_2\vec{e}_3)$$

$$x_1 x_2 \underbrace{\vec{e}_1 \times \vec{e}_1}_{\vec{0}} + y_1 y_2 \underbrace{\vec{e}_2 \times \vec{e}_2}_{\vec{0}} + z_1 z_2 \underbrace{\vec{e}_3 \times \vec{e}_3}_{\vec{0}} +$$

$$x_1 y_2\, \vec{e}_1 \times \vec{e}_2 + x_1 z_2\, \vec{e}_1 \times \vec{e}_3$$

$$y_1 x_2 \underbrace{\vec{e}_2 \times \vec{e}_1}_{-\vec{e}_1 \times \vec{e}_2} \quad z_1 x_2 \overbrace{\vec{e}_3 \times e_1}^{-\vec{e}_1 \times e_3} + y_1 z_2\, \vec{e}_2 \times \vec{e}_3 + z_1 y_2 \underbrace{\vec{e}_3 \times \vec{e}_2}_{-\vec{e}_2 \times \vec{e}_3}$$

$$= (x_1 y_2 - y_1 x_2) \underbrace{\vec{e}_1 \times \vec{e}_2}_{\vec{e}_3} + (x_1 z_2 - z_1 x_2) \underbrace{\vec{e}_1 \times \vec{e}_3}_{-\vec{e}_2}$$

$$+ (y_1 z_2 - z_1 y_2) \underbrace{\vec{e}_2 \times \vec{e}_3}_{\vec{e}_1}$$

$$= \left[\begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}, - \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix}, \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \right]$$

$$= \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

$$\begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{vmatrix} \end{array} = \overset{\vec a}{[1,2,3]} \times \overset{\vec b}{[3,2,1]}$$

$$= \left[\begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix}, -\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} \right]$$

$$= [\,2-6\,,\, -(1-9)\,,\, 2-6\,]$$

$$= [-4, 8, -4]$$

$$= \underline{4[-1, 2, -1]}$$

sjekker at $\vec a \times \vec b$ står vinkelrett på $\vec a$ og $\vec b$

$$4[-1,2,-1] \cdot [1,2,3] = 4(-1+4-3) = 0$$

$$4[-1,2,-1] \cdot [3,2,1] = 4(-3+4-1) = 0.$$

$$[4,8,20] \times [-12, 6, 18] =$$

$$4[1,2,5] \times (6[-2,1,3])$$

$$= 4 \cdot 6\, [1,2,5] \times [-2,1,3]$$

$$4 \cdot 6 \begin{vmatrix} \vec e_1 & \vec e_2 & \vec e_3 \\ 1 & 2 & 5 \\ -2 & 1 & 3 \end{vmatrix}$$

$$24\left[\begin{vmatrix}2&5\\1&3\end{vmatrix}, -\begin{vmatrix}1&5\\-2&3\end{vmatrix}, \begin{vmatrix}1&2\\-2&1\end{vmatrix}\right]$$

$$= 24[6-5, -(3+10), 1+4]$$

$$= 24[1, -13, 5]$$

sjekker at x-prod. står vinkelrett på

$[1,2,5]$ og $[-2,1,3]$

$*\ [1,2,5]\cdot[1,-13,5] = 1-26+25 = 0$ ✓

$[-2,1,3]\cdot[1,-13,5] = -2-13+15 = 0$ ✓

$$[4,8,20]\times\underbrace{[5,8,19]}_{[4,8,20]+[1,0,-1]}$$

$$= \underbrace{[4,8,20]\times[4,8,20]}_{\vec{0}} + 4[1,2,5]\times[1,0,-1]$$

$$= 4\begin{vmatrix}\vec{e}_1 & \vec{e}_2 & \vec{e}_3\\ 1 & 2 & 5\\ 1 & 0 & -1\end{vmatrix} = 4\left[\begin{vmatrix}2&5\\0&-1\end{vmatrix}, -\begin{vmatrix}1&5\\1&-1\end{vmatrix}, \begin{vmatrix}1&2\\1&0\end{vmatrix}\right]$$

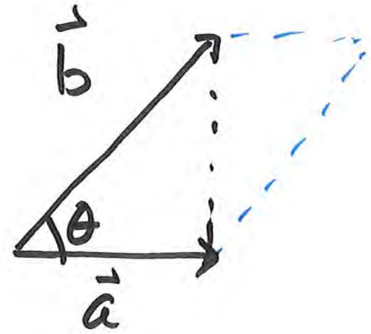
$$= 4[-2, -(-1-5), -2] = 4[-2, 6, -2]$$

$$= \underline{8[-1, 3, -1]}$$

Finn arealet til trekanten utspent av vektorene

$\vec{a} = [-1, 3, 4]$

og $\vec{b} = [5, 5, -10]$.



Areal $A = \frac{1}{2} \underbrace{|\vec{a}| \cdot |\vec{b}| \sin\theta}_{|\vec{a} \times \vec{b}|}$.

$$\vec{a} \times \vec{b} = [-1,3,4] \times [5,5,-10]$$
$$= 5([-1,3,4] \times [1,1,-2])$$
$$= 5 \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ -1 & 3 & 4 \\ 1 & 1 & -2 \end{vmatrix}$$
$$= 5\left[\begin{vmatrix} 3 & 4 \\ 1 & -2 \end{vmatrix}, -\begin{vmatrix} -1 & 4 \\ 1 & -2 \end{vmatrix}, \begin{vmatrix} -1 & 3 \\ 1 & 1 \end{vmatrix} \right]$$
$$= 5[-6-4, -(2-4), -1-3]$$
$$= 5[-10, 2, -4] = \underline{10[-5, 1, -2]}$$

$$|\vec{a} \times \vec{b}| = 10\sqrt{(-5)^2 + 1^2 + (-2)^2}$$
$$= 10\sqrt{25+1+4} = 10\sqrt{30}$$

Arealet til trekanten er lik $\frac{1}{2}|\vec{a} \times \vec{b}|$

$$= \underline{\underline{5\sqrt{30}}}$$