24feb
25

Determinanter 12F

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = a\cdot d - b\cdot c$$

Tar inn to vektorer og gir en skalar

$$\begin{vmatrix} [a, b] \\ [c\ d] \end{vmatrix} = \det([a,b],[c,d])$$

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = \begin{vmatrix} \vec{e}_1 \\ \vec{e}_2 \end{vmatrix} = 1\cdot 1 - 0\cdot 0 = 1$$

$$\begin{vmatrix} \vec{e}_2 \\ \vec{e}_1 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 0\cdot 0 - 1\cdot 1 = -1.$$

$$\begin{vmatrix} c & d \\ a & b \end{vmatrix} = c\cdot b - ad = -(ad - bc) = -\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

<u>antisymmetrisk.</u>

matrise $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$
2×2

$$\begin{bmatrix} 1 & 2 & 3 \\ 5 & -1 & 4 \end{bmatrix}$$ 2×3-matrise

$$= \begin{bmatrix} [1, 2, 3] \\ [5, -1, 4] \end{bmatrix} = \left[\begin{bmatrix} 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right]$$

2 radvektorer

3 søylevektorer

$\begin{bmatrix} \ddots & \\ & \ddots \end{bmatrix}$ — hoveddiagonalen

Bytte om rade- og søyle-vektorer. Det kalles å transponere matrise.

$$\begin{bmatrix} [a, b] \\ [c, d] \end{bmatrix}^{trans.} = \left[\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix} \right]$$

"Reflekterer om hoveddigonalen"

$$\begin{bmatrix} 1 & 2 & 3 \\ 5 & -1 & 4 \end{bmatrix}^{trans} = \left[\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ -1 \\ 4 \end{bmatrix} \right] = \begin{bmatrix} 1 & 5 \\ 2 & -1 \\ 3 & 4 \end{bmatrix}.$$

$$(2\times 3 \text{ matrise})^{trans} = 3\times 2\text{-matrise}.$$

Determinanten er uendret under transponering

$$\left| \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{trans} \right| = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - b\cdot c$$
$$= \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$\begin{vmatrix} \vec{a} \\ \vec{a} \end{vmatrix} = -\begin{vmatrix} \vec{a} \\ \vec{a} \end{vmatrix}$ bytter de to like vektorene

så $2\begin{vmatrix} \vec{a} \\ \vec{a} \end{vmatrix} = 0$ og derfor $\underline{\begin{vmatrix} \vec{a} \\ \vec{a} \end{vmatrix} = 0}$

Determinanten er lineær i begge rad- og søylevektorene dvs. respekterer sum og skalering.

$$\begin{vmatrix} k\vec{a} \\ \vec{b} \end{vmatrix} = k \begin{vmatrix} \vec{a} \\ \vec{b} \end{vmatrix} \quad \text{og} \quad \begin{vmatrix} \vec{a}_1 + \vec{a}_2 \\ \vec{b} \end{vmatrix} = \begin{vmatrix} \vec{a}_1 \\ \vec{b} \end{vmatrix} + \begin{vmatrix} \vec{a}_2 \\ \vec{b} \end{vmatrix} \text{ etc.}$$

$$\begin{vmatrix} k[a\ b] \\ [c\ d] \end{vmatrix} = \begin{vmatrix} ka & kb \\ c & d \end{vmatrix} = ka\cdot d - kb\cdot c = k(ad - bc)$$

$$= k \begin{vmatrix} a & b \\ c & d \end{vmatrix} \checkmark$$

$$\begin{vmatrix} [a_1, b_1] + [a_2, b_2] \\ [c, d] \end{vmatrix} = \begin{vmatrix} a_1 + a_2, & b_1 + b_2 \\ c & d \end{vmatrix} = (a_1 + a_2)d - (b_1 + b_2)\cdot c$$

$$(a_1 \cdot d - b_1 \cdot c) - (a_2 d - b_2 c) = \begin{vmatrix} a_1 & b_1 \\ c & d \end{vmatrix} + \begin{vmatrix} a_2 & d_2 \\ c & d \end{vmatrix} \checkmark$$

eks

$$\begin{vmatrix} 9 & 27 \\ 15 & 25 \end{vmatrix} = \begin{vmatrix} 9[1,3] \\ 5[3,5] \end{vmatrix} = 9\cdot 5 \begin{vmatrix} 1 & 3 \\ 3 & 5 \end{vmatrix}$$

$$= 9\cdot 5\,(1\cdot 5 - 3\cdot 3) = 9\cdot 5(-4) = \underline{-180}$$

$$\begin{vmatrix} 13 & 27 \\ 15, & 25 \end{vmatrix} = \begin{vmatrix} [15,25] + [-2,2] \\ [15, 25] \end{vmatrix} = \underbrace{\begin{vmatrix} 15, & 25 \\ 15 & 25 \end{vmatrix}}_{0} + \begin{vmatrix} -2, & 2 \\ 15 & 25 \end{vmatrix}$$

$$= \begin{vmatrix} -2 & 2 \\ 15 & 25 \end{vmatrix}$$

$$= 2\cdot 5 \begin{vmatrix} -1 & 1 \\ 3 & 5 \end{vmatrix} = 10\,(-1\cdot 5 - 1\cdot 3)$$

$$= \underline{-80}.$$

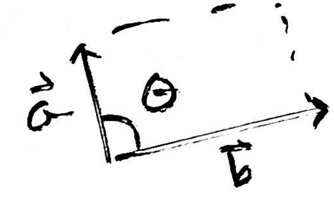
oppg. Regn ut $\det\left(\begin{bmatrix} 7 & 24 \\ 21 & 8 \end{bmatrix}\right) = \begin{vmatrix} 7 & 24 \\ 21 & 8 \end{vmatrix}$ (benytt gjerne linearitet)

$$= \left| 7\begin{bmatrix} 1 \\ 3 \end{bmatrix}, 8\begin{bmatrix} 3 \\ 1 \end{bmatrix} \right| = 7\cdot 8 \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} = 7\cdot 8(1\cdot 1 - 3\cdot 3)$$

$$= -7\cdot 8\cdot 8 = -7(64) = (-5-2)64$$

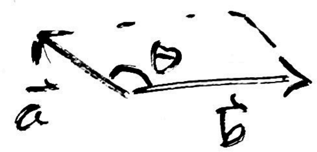
$$= -320 + (-128) = \underline{-448}$$

$$\text{abs}\left(\begin{vmatrix}\vec{a}\\ \vec{b}\end{vmatrix}\right) = |\vec{a}||\vec{b}| \cdot \sin\theta|$$



absoluttverdien

Dette er arealet til parallellogrammet utspent av $\vec{a}$ og $\vec{b}$.



$0 \leq \theta \leq 180°$

så $\sin\theta \geq 0$.

Bare lik 0 når $\theta = 0°$ eller $180°$.

Hvorfor er det slik?

$$[a, b] \cdot [c, d] = ac + bd = |\vec{a}||\vec{b}|\cos\theta$$

skalarproduktet

$\sin^2\theta + \cos^2\theta = 1$ for alle θ.

$$(|\vec{a}||\vec{b}|)^2 - (|\vec{a}||\vec{b}|\cos\theta)^2 = |\vec{a}|^2|\vec{b}|^2(1-\cos^2\theta)$$
$$= |\vec{a}|^2|\vec{b}|^2\sin^2\theta$$

Vi regner ut

$$|\vec{a}|^2|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2 = (a^2+b^2)(c^2+d^2) - (ac+bd)^2$$
$$= a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 - a^2c^2 - b^2d^2 - 2acbd$$
$$= a^2d^2 + b^2c^2 - 2(ad)(bc) = (ad-bc)^2$$

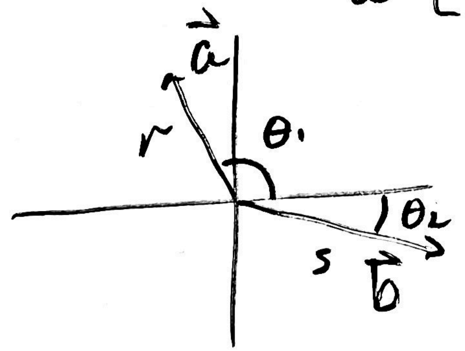
Så $|ad-bc| = |\vec{a}||\vec{b}|\sin\theta$ $\square$

$\left|\begin{matrix} a & b \\ c & d \end{matrix}\right|$

Alternativt bevis.

$\vec{a} = [r\cos\theta_1, r\sin\theta_1]$

$\vec{b} = [s\cos\theta_2, s\sin\theta_2]$



$$\left|\begin{matrix}\vec{a}\\ \vec{b}\end{matrix}\right| = rs \begin{vmatrix} \cos\theta_1 & \sin\theta_1 \\ \cos\theta_2 & \sin\theta_2 \end{vmatrix}$$

$$= rs\left(\cos\theta_1 \sin\theta_2 - \cos\theta_2 \sin\theta_1\right)$$

Benytte at $\sin(-\theta_1) = -\sin\theta_1$

$\cos(-\theta_1) = \cos\theta_1$

$$\left|\begin{matrix}\vec{a}\\ \vec{b}\end{matrix}\right| = rs\left(\cos(-\theta_1)\sin\theta_2 + \cos\theta_2 \sin(-\theta_1)\right)$$

$$= \underline{rs \sin(\theta_2 - \theta_1)}$$

$|\theta_2 - \theta_1|$ vinkelen mella $\vec{a}, \vec{b}$.

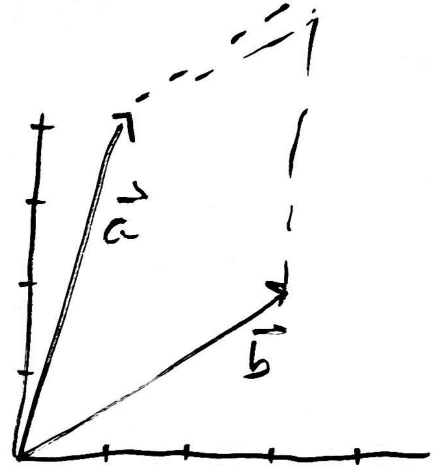
positiv hvis $180° > \theta_2 - \theta_1 > 0$

negativ hvis $-180° < \theta_2 - \theta_1 < 0$

$\square$

$\vec{a}, \vec{b}$ er parallelle $\iff$ $\left|\begin{matrix}\vec{a}\\ \vec{b}\end{matrix}\right| = 0$

Finn arealet til parallellogrammet utspent av
$\vec{a} = [1, 4]$ og $\vec{b} = [3, 2]$.



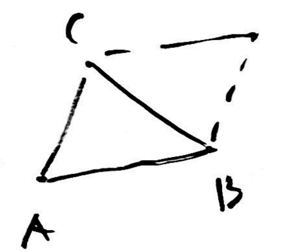
$$\left\| \begin{matrix} \vec{a} \\ \vec{b} \end{matrix} \right\| = \left| \det \begin{pmatrix} \vec{a} \\ \vec{b} \end{pmatrix} \right|$$

$$= \text{abs}\left(\begin{vmatrix} 1 & 4 \\ 3 & 2 \end{vmatrix} \right) = \text{abs}(1 \cdot 2 - 3 \cdot 4)$$

$$= \text{abs}(-10) = \underline{\underline{10}}$$

Tre punkt $A(1,2)$ $B(3,5)$ og $C(-1,-3)$ i planet.

Finn arealet til $\triangle ABC$.



$\frac{1}{2}$ areal parallellogrammet utspent av $\vec{AB}$ og $\vec{AC}$

$$\vec{AB} = \vec{OB} - \vec{OA} = [3,5] - [1,2] = [2,3]$$

$$\vec{AC} = \vec{OC} - \vec{OA} = [-1,-3] - [1,2] = [-2,-5].$$

$$\frac{1}{2}\left|\begin{matrix}\vec{AB}\\ \vec{AC}\end{matrix}\right| = \text{abs}\left(\frac{1}{2}\begin{vmatrix}2 & 3\\ -2 & -5\end{vmatrix}\right) = \left|\frac{1}{2}(-1)\cdot 2\begin{vmatrix}1 & 3\\ 1 & 5\end{vmatrix}\right|$$

$$= \text{abs}\left(\begin{vmatrix}1 & 3\\ 1 & 5\end{vmatrix}\right) = |1\cdot 5 - 3\cdot 1| = \underline{\underline{2}}$$

Oppg. Finn arealet til parallellogrammet utspent av $\vec{a} = [-1, 4]$ og $\vec{b} = [3, 5]$.

$$\text{abs}\left(\begin{vmatrix}-1 & 4\\ 3 & 5\end{vmatrix}\right) = \text{abs}\left((-1)\cdot 5 - 3\cdot 4\right) = |-5-12| = \underline{17}$$

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}$$

lineært likningssystem med variable x og y.

$$x\begin{bmatrix}a\\ c\end{bmatrix} + y\begin{bmatrix}b\\ d\end{bmatrix} = \begin{bmatrix}e\\ f\end{bmatrix}$$

Ta determinanten med $\begin{bmatrix}b\\ d\end{bmatrix}$

$$\det\left(x\begin{bmatrix}a\\c\end{bmatrix}+y\begin{bmatrix}b\\d\end{bmatrix},\begin{bmatrix}b\\d\end{bmatrix}\right) = \det\left(\begin{bmatrix}e\\f\end{bmatrix},\begin{bmatrix}b\\d\end{bmatrix}\right)$$

$$x\det\begin{pmatrix}a & b\\ c & d\end{pmatrix} + y\underbrace{\det\begin{pmatrix}b & b\\ d & d\end{pmatrix}}_{0} = \det\begin{pmatrix}e & b\\ f & d\end{pmatrix}$$

så $x = \dfrac{\begin{vmatrix}e & b\\ f & d\end{vmatrix}}{\begin{vmatrix}a & b\\ c & d\end{vmatrix}}$ og tilsvarende $y = \dfrac{\begin{vmatrix}a & e\\ c & f\end{vmatrix}}{\begin{vmatrix}a & b\\ c & d\end{vmatrix}}$.

Kramers regel.

3×3 determinanter

$$\begin{vmatrix}x_{1,1} & x_{1,2} & x_{1,3}\\ x_{2,1} & x_{2,2} & x_{2,3}\\ x_{3,1} & x_{3,2} & x_{3,3}\end{vmatrix} = x_{11}\begin{vmatrix}x_{22} & x_{23}\\ x_{32} & x_{33}\end{vmatrix} - x_{12}\begin{vmatrix}x_{21} & x_{23}\\ x_{31} & x_{33}\end{vmatrix} + x_{13}\begin{vmatrix}x_{21} & x_{22}\\ x_{31} & x_{32}\end{vmatrix}$$

eksempel

$$\begin{vmatrix} 1 & 4 & 0 \\ -1 & 2 & 3 \\ 5 & 1 & 2 \end{vmatrix}$$

$$= 1 \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} - 4 \begin{vmatrix} -1 & 3 \\ 5 & 2 \end{vmatrix} + 0 \cdots$$

$$= (4-3) - 4(-2-15) = 1 - 4(-17)$$

$$= \underline{69}$$

ØVING

$$\begin{vmatrix} x & y \\ z & w \end{vmatrix} = xw - yz.$$

$$\begin{vmatrix} a & d \\ b & c \end{vmatrix} = ac - bd.$$

Når er $[s^2, s^2-1]$ og $[2,1]$ parallelle?

$$\begin{vmatrix} 2 & 1 \\ s^2 & s^2-1 \end{vmatrix} = 0$$

$$2(s^2-1) - s^2 = 0$$

$$s^2 - 2 = 0 \iff s^2 = 2$$

Nå $s = \pm\sqrt{2}$.

oppgave Når er $[s, s^2-1]$ og $[2,1]$ parallelle?

$$\iff \begin{vmatrix} 2 & 1 \\ s & s^2-1 \end{vmatrix} = 2(s^2-1) - s = 2s^2 - 2 - s = 0$$

$$s = \frac{1 \pm \sqrt{1-4(2)(-2)}}{4} = \underline{\frac{1 \pm \sqrt{17}}{4}}$$

Når er vektorene $[s, s^2-1]$ og $[2,1]$ ortogonale?

$$\Leftrightarrow \quad [s, s^2-1]\cdot[2,1] = 0$$

$$2s + s^2 - 1 = 0$$

$$(s+1)^2 - 2 = 0 \qquad s+1 = \pm\sqrt{2}$$

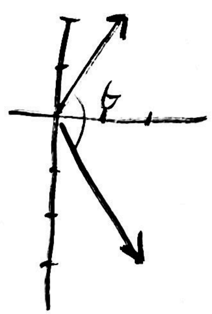
$$\underline{s = -1 \pm \sqrt{2}}$$

$\vec{a}$ og $\vec{b}$ er <u>ortogonale</u> hvis og bare hvis $\vec{a}\bullet\vec{b} = 0$

$= |\vec{a}||\vec{b}|\cos(\theta)$

$\vec{a}$ eller $\vec{b}$ er $\vec{0}$-vektoren. eller $\vec{a}, \vec{b} \neq \vec{0}$ og vinkelrette på hverandre.

$$\overset{\vec{a}}{[1,2]}\cdot\overset{\vec{b}}{[2,-3]} = 1\cdot 2 + 2(-3) = -4 \neq 0 \quad \text{ikke ortogonale.}$$



$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{-4}{\sqrt{5}\sqrt{13}}$$

$$= \frac{-4}{\sqrt{65}} \quad (\sim \tfrac{-1}{2}$$

$$\theta \sim \underline{119.74°}$$

Når er vektorene $\vec{a} = [s^2, s]$ og $\vec{b} = [1-s, 2]$ 1) parallelle 2) ortogonale ?

1. parallelle: $\left|\begin{matrix}\vec{a} \\ \vec{b}\end{matrix}\right| = \left|\begin{matrix} s^2, & s \\ 1-s, & 2\end{matrix}\right| = s\left|\begin{matrix} s, & 1 \\ 1-s & 2\end{matrix}\right| = s(2s - (1-s))$

$= s(3s-1) = 0$

når

$\underline{s=0}$ og $\underline{s=1/3}$

2. ortogonal $\quad \vec{a} \cdot \vec{b} = [s^2, s] \cdot [1-s, 2] = 0$

$= s[s,1] \cdot [1-s, 2]$

$= s(s(1-s) + 1 \cdot 2) = s(-s^2 + s + 2)$

skulle vært -2!

$= -s(s^2 - s - 1) = 0$

$\underline{s=0}$ og $s = \dfrac{1 \pm \sqrt{1+4}}{2} = \underline{\dfrac{1 \pm \sqrt{5}}{2}}$.

(mer detaljer: $s^2 - s - 1 = 0 \quad a=1, b=-1, c=-1$

$s = \dfrac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2 \cdot 1} = \dfrac{1 \pm \sqrt{5}}{2}$)

$s=0$ og $s^2-s-2=0$

$$s=\frac{1\pm\sqrt{1-(-8)}}{2}=\frac{1\pm3}{2}$$

$s=-1$

$s=\frac{1+3}{2}=2$

ortogonale når $\underline{s=0}$, $\underline{s=-1}$ og $\underline{s=2}$

Finn

$$\begin{array}{ccc} + & - & + \end{array}$$

$$\begin{vmatrix} 6 & -3 & 12 \\ 4 & 8 & 12 \\ 10 & 25 & 15 \end{vmatrix} = \begin{vmatrix} 3[2,-1,4] \\ 4[1,2,3] \\ 5[2,5,3] \end{vmatrix} = 3\cdot4\cdot5 \begin{vmatrix} 2 & -1 & 4 \\ 1 & 2 & 3 \\ 2 & 5 & 3 \end{vmatrix}$$

$$=60\left(2\begin{vmatrix}2&3\\5&3\end{vmatrix}-(-1)\begin{vmatrix}1&3\\2&3\end{vmatrix}+4\left(\begin{vmatrix}1&2\\2&5\end{vmatrix}\right)\right)$$

$$=60\left(2\cdot3\begin{vmatrix}2&1\\5&1\end{vmatrix}+3\begin{vmatrix}1&1\\2&1\end{vmatrix}+4\begin{vmatrix}1&2\\2&5\end{vmatrix}\right)=$$

$$60\left(6(2-5)+3(1-2)+4(5-2^2)\right)$$

$$=60(-18-3+4)=60(-17)$$

$$=-(600+420)=\underline{-1020}$$