5 feb
25

Diverse – derivasjon.

Minner dere på informasjonsmøte torsdag 27. februar. Påmelding.

Vi har sett at $(x^r)' = r x^{r-1}$ når r naturlig tall.

$r=0$ $\quad (x^0)' = (1)' = 0$ så lik $0 \cdot x^{0-1} = \frac{0}{x}$ ✓ (når $x \neq 0$)

$n \geq 1$ naturlig : $\quad (x^{-n})' = ((x^n)^{-1})' = \left(\frac{1}{x^n}\right)'$

(benytter at $\left(\frac{1}{x}\right)' = \lim_{h\to 0} \frac{\left(\frac{1}{x+h} - \frac{1}{x}\right)}{h} = \lim_{h \to 0} \frac{\left(\frac{x-(x+h)}{(x+h)x}\right)}{h}$

$= \lim_{h\to 0} \frac{-1}{x(x+h)} = \frac{-1}{x^2}$ så ok for $n=1$)

Benytter kjerneregelen : $\quad \left(\frac{1}{x^n}\right)' = \frac{-1}{(x^n)^2} \cdot \underbrace{(x^n)'}_{n x^{n-1}} = \frac{-n x^{n-1}}{x^{2n}}$

$= -n x^{n-1-2n} = -n x^{-2n+n-1}$

$= \underline{-n x^{-n-1}}$ ✓

så $(x^r)' = r x^r$ for $r \in \mathbb{Z}$

$\sqrt[n]{x} = x^{1/n}$ $\qquad$ $\sqrt[n]{x^m} = (x^m)^{1/n} = \underline{x^{m/n}}$ $\qquad$ $n>0$, $x>0$

Deriverer: $\left(\sqrt[n]{x^m}\right)^n = x^m$

$$n\left(\sqrt[n]{x^m}\right)^{n-1} \cdot \left(\sqrt[n]{x^m}\right)' = m x^{m-1}$$

så $$\left(x^{m/n}\right)' = \frac{m x^{m-1}}{n \left(x^{m/n}\right)^{n-1}}$$

$$= \frac{m}{n} x^{m-1} \Big/ \underbrace{x^{\frac{m(n-1)}{n}}}_{x^m \cdot x^{-m/n}}$$

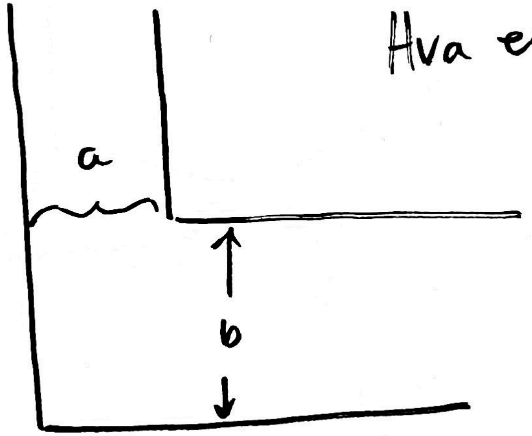
$$= \frac{m}{n} x^{m-1} \cdot x^{-m} \cdot x^{m/n}$$

$$= \frac{m}{n} x^{(m/n)-1} \checkmark$$

$$\left(x^{m/n}\right)' = \frac{m}{n} x^{(m/n)-1}$$

så $\underline{(x^r)' = r x^{r-1} \text{ for } r \in \mathbb{Q}}$

Formelen er gyldig for alle $r \in \mathbb{R}$.

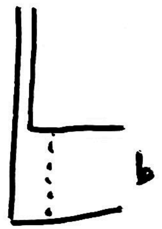


Hva er lengden til den lengste stokken som kommer forbi hjørne?

$$\left(a^{2/3} + b^{2/3}\right)^{3/2}$$

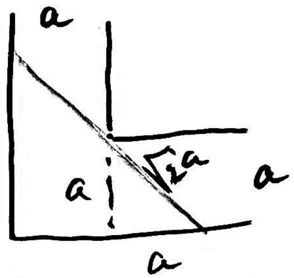
Løsningen er vist i et notat på nettsiden til kurset.

$a=0$ ✓

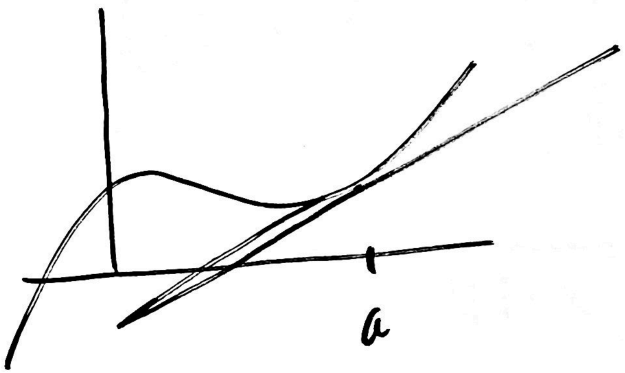


$a=b$

$2\sqrt{2}\,a$



$$\left(a^{2/3} + a^{2/3}\right)^{3/2}$$
$$= \left(2a^{2/3}\right)^{3/2}$$
$$= 2^{3/2}\, a^{2/3 \cdot 3/2}$$
$$= 2\sqrt{2}\, a \quad ✓$$



tangentlinjen i $(a, f(a))$

$$y = f'(a)(x-a) + f(a)$$

eks. Tangentlinjen til $f = \sin(2x-1)$ i $x = 2$:

$$f' = \cos(2x-1)\cdot(2x-1)'$$
$$= 2\cos(2x-1)$$
$$f'(2) = 2\cos(3)$$

$$y = 2\cos(3)(x-2) + \sin(3).$$

Verdiene på tangentlinjen er lineære tilnærminger til $f(x)$ for x nær a.

$$(\sqrt{1+x})' = \left((1+x)^{1/2}\right)' = \frac{1}{2}(1+x)^{1/2-1}\underbrace{(1+x)'}_{1}$$

$$= \frac{1}{2\sqrt{1+x}}.$$

$$\sqrt{1+x} \sim \underset{f'(0)}{\frac{1}{2}}x + \underset{f(0)}{1} = 1 + \frac{x}{2}.$$

x liten er $\sqrt{1+x} \sim 1 + \frac{x}{2}$.

$$\sqrt{10} = \sqrt{9+1} = \sqrt{9(1+\tfrac{1}{9})} = 3\sqrt{1+\tfrac{1}{9}}.$$

$$\sim 3\left(1 + \frac{1}{2}\cdot\frac{1}{9}\right) = 3 + \frac{1}{6}$$

Vi kan tilnærme en funksjon $f(x)$ rundt $x=a$ med et polynom av grad n $P_n(x)$ slik at $(f(x) - P_n(x))^{(m)}(a) = 0$ for $m \leq n$.

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2} + \cdots + \frac{f^{(n)}(a)}{1\cdot 2\cdot 3\cdots n}(x-a)^n.$$

$$\left((1+x)^{1/2}\right)'' = \left(\frac{1}{2}(1+x)^{-1/2}\right)' = \frac{1}{2}\left(-\frac{1}{2}\right)(1+x)^{-3/2}.$$

Så til 2. orden $\sqrt{1+x} \sim 1 + \frac{x}{2} - \frac{x^2}{8}$.

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{7!} + \frac{x^9}{9!} - + \cdots$$

Se hjemmesiden til kurset for en visualisering av dette.

$$\begin{cases} \dfrac{f(x+h)-f(x)}{h} - f'(x) & \overset{h\neq 0}{=} r(x,h) = r(h) \quad (\text{holder } x \text{ fast}) \\ 0 \quad \text{når } h=0 & \lim_{h\to 0} r(x,h) = 0. \end{cases}$$

$$f(x+h) - f(x) = (f'(x) + r(h))h$$

$$\underline{f(x+h) = f(x) + h(f'(x) + r(h))}$$

Kjerneregelen

$$u(x+h) = u(x) + h(u'(x) + s(h))$$

$$f(u(x+h)) = f(u(x) + h(u'(x) + s(h)))$$

$$= f(u(x)) + (h(u'(x) + s(h)))(f'(u(x)) + r(h(u'(x)+s(h))))$$

$$(f\circ u)'(x) = \lim_{h\to 0} \frac{f\,u(x+h) - f(u(x))}{h}$$

$$(f\circ u)'(x) = \lim_{h\to 0} \underbrace{\frac{h}{h}}_{1} \Big(u'(x) + \underbrace{s(h)}_{\to 0}\Big)\underbrace{\Big(f'(u(x)) + r\big(\underbrace{h(u'(x)+s(h))}_{\to 0}\big)\Big)}_{\to 0}$$

$$= u'(x)\cdot f'(u(x))$$

$$= \underline{f'(u(x))\,u'(x)}$$

Vi har vist kjerneregelen.

Eksponentfunksjoner

Potens a^r ← eksponent. ← grunntallet

x^r potensfunksjon

a^x eksponentfunksjon

$$(a^x)' = \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} = \lim_{h\to 0} \frac{a^x(a^h - 1)}{h}$$

$$= a^x \lim_{h\to 0} \frac{a^h - 1}{h}$$

så $(a^x)' = (\text{konstant}) \cdot a^x$. (hvis den er deriverbar)

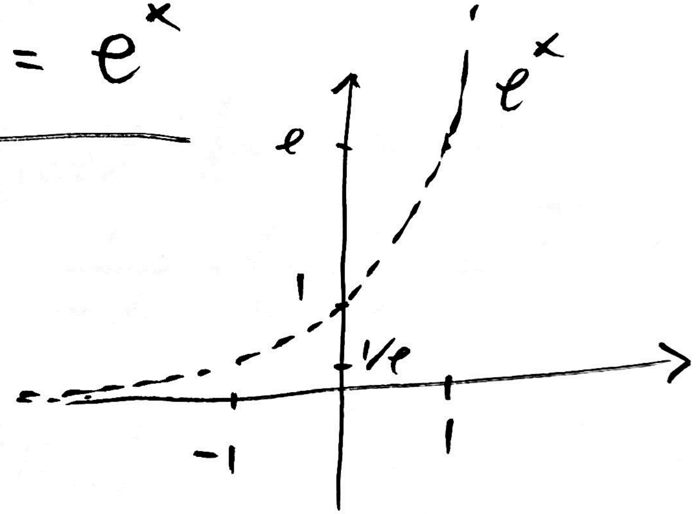
$\uparrow$

blir lik 1

når $a = e$ Euler tallet

2.718281828.... irrasjonalt.

$$\underline{(e^x)' = 1 \cdot e^x = e^x}$$



Inversfunksjoner er naturlig logaritme $\ln(x)$

$$e^{\ln(x)} = x \qquad x > 0$$

$$a = e^{\ln a} \quad \text{så} \quad a^x = (e^{\ln a})^x = e^{x \ln a}$$

$$(a^x)' = \underbrace{e^{x \cdot \ln a} \cdot (x \cdot \ln a)'}_{\text{kjerneregel}} = \underline{\ln(a) \cdot a^x}$$

$$(a^x)' = \ln(a) \cdot a^x$$

$$(2^x)' = \ln 2 \cdot 2^x \sim (0.693147\ldots)\, 2^x$$

$$(10^x)' = \ln 10 \cdot 10^x \sim (2.302585\ldots)\, 10^x$$

$$(\ln x)' = \frac{1}{x} \qquad x > 0$$

$\ln(-x)$ er definert for $x < 0$

$$(\ln(-x))' \underset{\text{kjerne}}{=} \frac{1}{-x} \cdot \underbrace{(-x)'}_{-1} = \frac{-1}{-x} = \frac{1}{x}.$$

kombinere dem: $\underline{(\ln |x|)' = \frac{1}{x}}$ $\qquad x \neq 0.$

$e^{\ln x} = x$ deriverer.

$$e^{\ln x} \cdot (\ln x)' = (x)' = 1 \quad \text{så} \quad (\ln x)' = \underline{\frac{1}{e^{\ln x}} = \frac{1}{x}} \checkmark$$

oppg Deriver

1) e^{-x^2}

$$\left(e^{-x^2}\right)' = e^{-x^2}\cdot\left(-x^2\right)' = \underline{-2x\,e^{-x^2}}$$

2) $\ln|\ln|x||$

$$\ln|\ln|x|| \overset{x\neq 0}{=} \frac{1}{\ln|x|}\cdot\left(\ln|x|\right)' = \underline{\frac{1}{x\ln|x|}}$$

3) $\left(\frac{1}{2}\right)^{1/x}$

$$\left(\left(e^{-\ln 2}\right)^{1/x}\right)' \overset{x\neq 0}{=} \left(e^{\left(-\ln 2\cdot\frac{1}{x}\right)}\right)'$$

$$= \left(\frac{1}{2}\right)^{1/x}\cdot\underbrace{\left(-\ln 2\cdot\frac{1}{x}\right)'}_{-\ln 2\cdot\frac{-1}{x^2}}$$

$$= \underline{\frac{\ln 2}{x^2}\left(\frac{1}{2}\right)^{1/x}}$$

$$(x^x)'$$

$$x^x = \left(e^{\ln x}\right)^x = e^{x \cdot \ln x}$$

$$\begin{aligned}(x^x)' &= \left(e^{x \ln x}\right)' \\ &= x^x \cdot (x \cdot \ln x)' \\ &= x^x\big(\underbrace{(x)'}_{1} \ln x + x \cdot \underbrace{(\ln x)'}_{1/x}\big)\end{aligned}$$

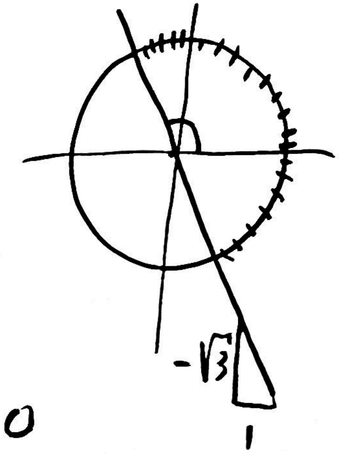
$$(x^x)' = \underline{x^x(\ln x + 1)}$$

Øving

Trig. ulikhet tilsvarende 11 a)

$$\sqrt{3}\cos x + \sin x > 0 \qquad [0, 2\pi\rangle$$

ikke med



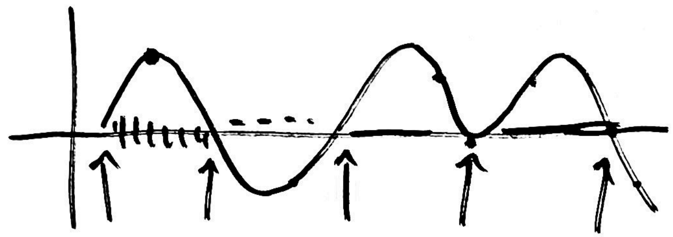
$$\sqrt{3}\cos x + \sin x = 0$$

$$\sin x = -\sqrt{3}\cos x \qquad \cos x \neq 0$$

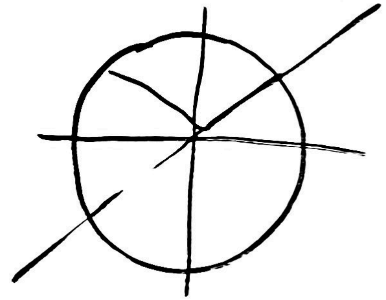
$$\tan x = -\sqrt{3} = -\frac{\sqrt{3}/2}{1/2}$$

$$x = \frac{2\pi}{3} + \pi \cdot n$$

Løsningene er $\underline{[0, \frac{2\pi}{3}\rangle \cup \langle \frac{5\pi}{3}, 2\pi\rangle}$



Sinus-kalkulator (Python)



x vinkel i radianer

Reduser til vinkel i $[0, 2\pi\rangle$

$[0, \pi\rangle$

$\begin{cases} 2\pi - x & \pi < x < 2\pi \\ x & \end{cases}$

Taylor polynom

$[0, \pi/2\rangle$

$\begin{cases} \pi - x & \frac{\pi}{2} < x < \pi \\ x & \end{cases}$

$\begin{cases} \pi/2 - x & \frac{\pi}{4} < x \\ & \end{cases}$ $\underline{[0, \pi/4\rangle}$

$\cos x$

$\sin x$

snur fortegnet til sin-verdien
hvis vi reflekterer om x-aksen

Kvotientregelen

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)\,g(x) - g'(x)\,f(x)}{g^2(x)}$$

produktregelen

$$\left(f\cdot\left(\frac{1}{g}\right)\right)' = f'\cdot\frac{1}{g} + f\cdot\underbrace{\left(\frac{1}{g}\right)'}_{\frac{-1}{g^2}\cdot g'}$$ kjerneregelen

$$= \frac{f'}{g} + f\frac{(-1)}{g^2}\cdot g' = \frac{f'\cdot g}{g^2} - \frac{f\cdot g'}{g^2}$$ ✓

$$\left(\frac{x}{e^x}\right)' = \left(x e^{-x}\right)' = (x)' e^{-x} + x\left(e^{-x}\right)'$$

$$= e^{-x} + x\left(-e^{-x}\right)$$

$$= \underline{(1-x)\,e^{-x}}$$

Behøver ikke benytte kvotientregelen på kvotienter.

$$\left(\frac{\sin x}{x^2}\right)' = \frac{(\sin x)' \cdot x^2 - \sin x \, (x^2)'}{(x^2)^2}$$

$$= \frac{x^2 \cos x - 2x \sin x}{x^4} = \underline{\frac{x \cos x - 2 \sin x}{x^3}}$$

Alternativt:

$$(x^{-2} \cdot \sin x)' = \underbrace{(x^{-2})'}_{-2x^{-3}} \sin x + (x^{-2}) \underbrace{(\sin x)'}_{\cos x}$$

$$= -2x^{-3} \sin x + x^{-2} \cos x \cdot \frac{x^3}{x^3}$$

$$= \frac{x \cos x - 2 \sin x}{x^3} .$$