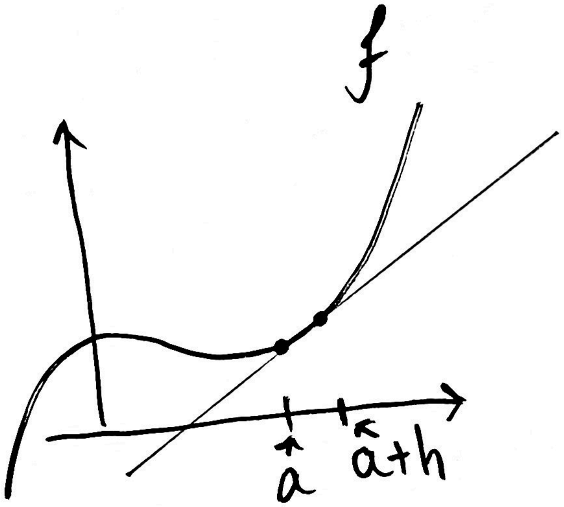
3 feb 25

11E Den deriverte til sin x, cos x og tan x.



delta

$\Delta f = f(a+h) - f(a)$

$\Delta x = (a+h) - a = h$

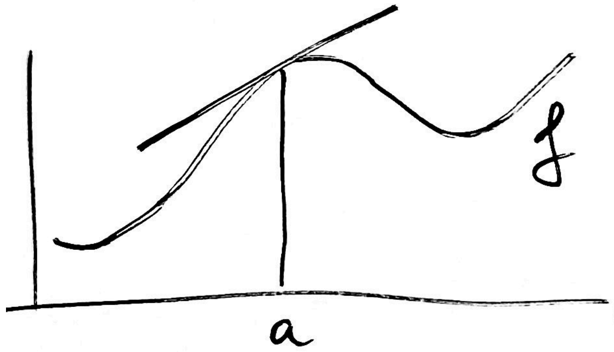
Endringsraten $\frac{\Delta f}{\Delta x}$ er lik stigningstallet til sekantlinjen gjennom $(a, f(a))$ og $(a+h, f(a+h))$

Momentan endringsrate "den deriverte" $f'(x) = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$ (behøver ikke eksistere)

Tangentlinjen til $f(x)$ for $x=a$ går gjennom $(a, f(a))$ og har stigningstall $f'(a)$

Sekantlinjene gjennom $(a, f(a))$ og $((a+h), f(a+h))$ nærmer seg tangentlinjen i $(a, f(a))$ når $h \to 0$.

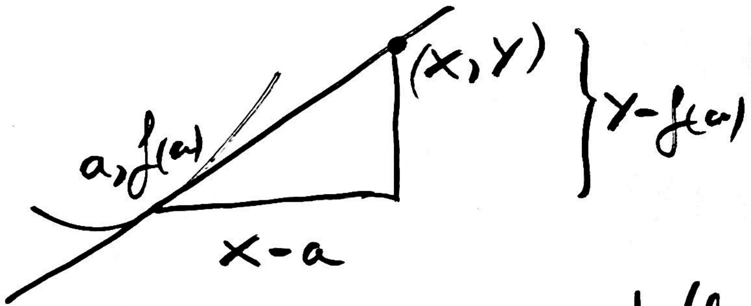
Tangentlinjer.



Tangentlinjen til $f(x)$ gjennom $(a, f(a))$ har stigningstall $f'(a)$.

$$y - f(a) = f'(a)(x-a)$$

$$y = f'(a)(x-a) + f(a)$$



($y = m(x - x_0) + y_0$ ← stigningstall; ett-punkts-formel)

$\frac{y - f(a)}{x - a}$ = stigningstallet til linjen $f'(a)$.

Numerisk er $\frac{f(a+h) - f(a-h)}{2h}$ nærmere $f'(a)$

enn $\frac{f(a+h) - f(a)}{h}$.

Derivasjon er lineær

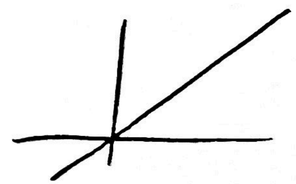
(definisjonen: $(kf)(x) = k \cdot f(x)$ ← konstant, $(f+g)(x) = f(x) + g(x)$)

$$(kf)'(x) = k \cdot (f'(x))$$
$$(f+g)'(x) = (f' + g')(x)$$

$$(x^r)' = r x^{r-1}$$ r reelt tall, $x > 0$

$f(x) = x$ definisjonen: $\frac{f(x+h) - f(x)}{h} = \frac{x+h-x}{h} = 1$

$$(x)' = \underline{1}.$$



$f(x) = \text{konstant} = k$ $\qquad \Delta f = 0$

$(k)' = 0$

k	

$f(x) = \frac{1}{x} = x^{-1}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{1}{x+h} - \frac{1}{x}\right) = \lim_{h \to 0} \frac{1}{h}\left(\frac{(x) - (x+h)}{(x+h)x}\right)$$

$$= \lim_{h \to 0} \frac{-h}{h(x+h)x} = \lim_{h \to 0} \frac{-1}{(x+h)x} = \frac{-1}{x^2} = (-1)x^{-2}$$

$$f(x) = \sqrt{x} = x^{1/2}$$

$$f'(x) = \lim_{h\to 0} \left(\frac{\sqrt{x+h} - \sqrt{x}}{h} \right) \cdot \frac{(\sqrt{x+h} + \sqrt{x})}{(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h\to 0} \underbrace{\left(\frac{(x+h) - x}{h} \right)}_{1} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

$$= \frac{1}{2} \cdot x^{-1/2}$$

$$(x^{1/2})' = \frac{1}{2} \cdot x^{\frac{1}{2}-1} \quad \ldots$$

$$(a+b)(a-b) = a^2 - b^2$$

Utvidet konjugatsetning:

$$(a^2 + ab + b^2)(a-b) = a^3 + a^2b + ab^2 - (a^2b + ab^2 + b^3)$$

$$= a^3 - b^3$$

n naturlig tall

$$(a^n + a^{n-1}b + a^{n-2}b^2 + \cdots + a^1b^{n-1} + b^n)(a-b)$$

$$= a^{n+1} - b^{n+1}$$

$$(x^n)' = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$$ ved generalisert konjugatsetning

$$= \lim_{h\to 0} \underbrace{\frac{(x+h-x)}{h}}_{1} \Big(\underbrace{(x+h)^{n-1} + (x+h)^{n-2}x + \dots + x^{n-1}}_{n \text{ ledd}} \Big)$$

$$= \underline{n \cdot x^{n-1}}.$$

$$(x^{17})' = 17x^{16}$$

$$(x) = (x^1) = 1 \cdot x^0 = 1$$

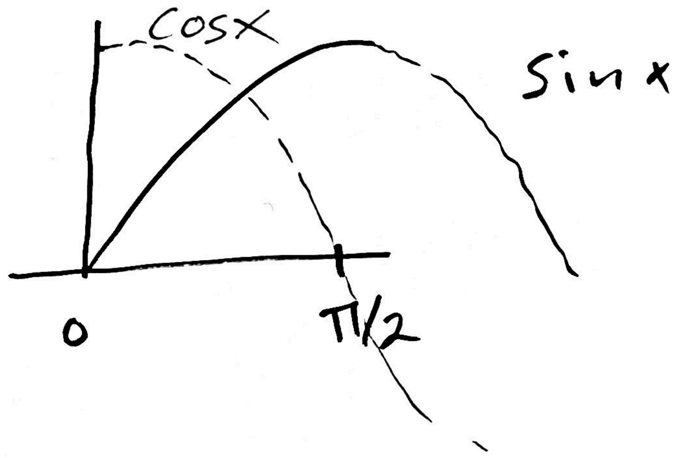
$$(x^3)' = 3x^2$$

$$(1)' = (x^0)' = 0 \cdot x^{-1} = 0$$

$(1)' = 0$ for alle x.

$$(\sin(x))' = \cos(x)$$

$$(\cos(x))' = -\sin(x)$$



$$(\sin x)' = \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

Addisjonsformel for sin:

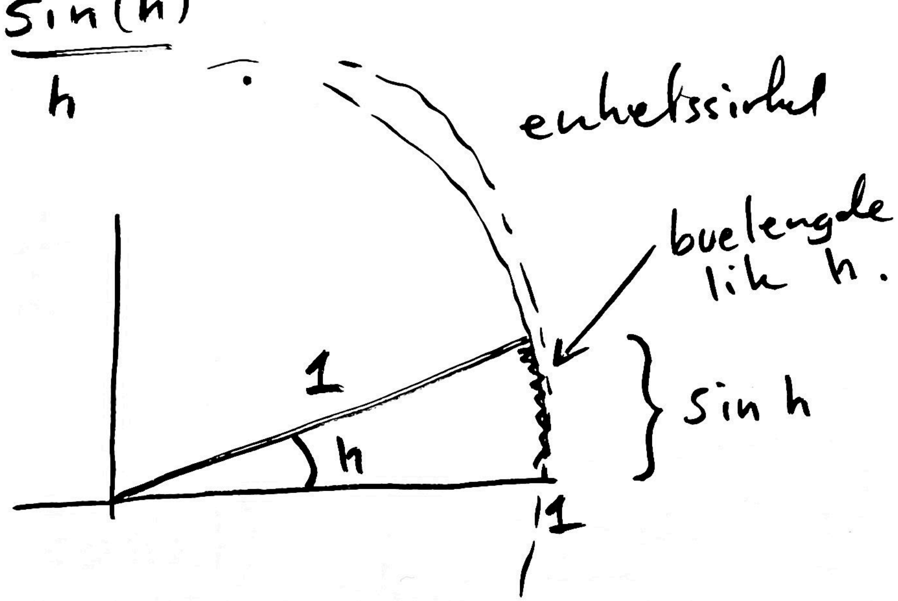
$$\sin(x+h) = \sin x \cdot \cos(h) + \sin(h) \cdot \cos x$$

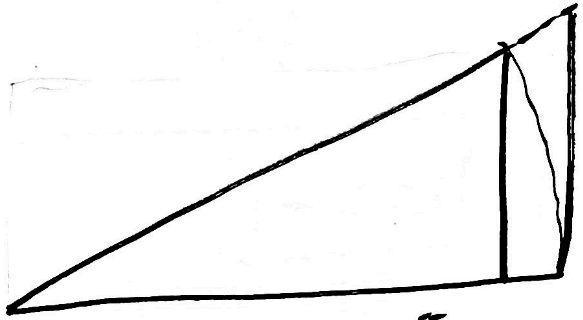
$$(\sin x)' = \lim_{h\to 0} \sin x \frac{\cos(h)-1}{h} + \lim_{h\to 0} \cos x \frac{\sin(h)}{h}$$

$$= \sin x \lim_{h\to 0} \frac{\cos(h)-1}{h} + \cos x \lim_{h\to 0} \frac{\sin(h)}{h}$$

$$\frac{\sin(-h)}{-h} = \frac{-\sin(h)}{-h} = \frac{\sin(h)}{h}.$$

Tilstrekkelig å finne $\lim_{h\to 0^+} \frac{\sin(h)}{h}$.





"areal liten Δ" ≤ areal sirkelsegment ≤ areal "stor Δ"

$$\frac{1}{2}\cos h \cdot \sin h \leq \frac{1}{2}\cdot h \leq \frac{1}{2}\cdot 1 \cdot \frac{\sin h}{\cos h}$$

$$\cos(h)\sin(h) \leq h \leq \frac{\sin(h)}{\cos(h)} \qquad \frac{\pi}{2} > h > 0$$

$$\cos(h) \leq \frac{\sin(h)}{h} \leq \frac{1}{\cos(h)} \qquad 0 < h < \frac{\pi}{2}.$$

Siden $\lim_{h\to 0} \cos(h) = 1 = \lim_{h\to 0} \frac{1}{\cos(h)}$, så $\lim_{h\to 0} \frac{\sin(h)}{h} = 1$

Skal vise:

$$\lim_{h\to 0} \frac{1-\cos h}{h} = 0,$$

$$\lim_{h\to 0} \frac{1-\cos(h)}{h^2} = \frac{1}{2}.$$

$$\lim_{h\to 0} \left(\frac{1-\cos(h)}{h^2}\right) \frac{(1+\cos(h))}{(1+\cos(h))} = \lim_{h\to 0} \frac{\overbrace{(1-\cos^2(h))}^{\sin^2(h)}}{h^2} \cdot \frac{1}{1+\cos(h)}$$

$$= \lim_{h\to 0} \left(\frac{\sin h}{h}\right)^2 \cdot \frac{1}{1+\cos(h)} = 1^2 \cdot \frac{1}{1+1} = \frac{1}{2}.$$

Det følger da at $\lim_{h\to 0} \frac{1-\cos h}{h} = \lim_{h\to 0} \frac{1-\cos h}{h^2} \cdot h$

$$= \frac{1}{2} \cdot 0 = 0.$$

Vi konkluderer at $\sin(x)' = \cos x \underbrace{\lim_{h\to 0} \frac{\sin(h)}{h}}_{1} + \sin x \underbrace{\lim_{h\to 0} \frac{\cos h - 1}{h}}_{0}$

$$(\sin x)' = \cos(x)\cdot 1 + \sin(x)\cdot 0$$
$$= \cos(x).$$

Kjerneregel

$$\big(f(u(x))\big)' = f'(u(x))\cdot u'(x)$$

$$\cos x = \sin\left(\frac{\pi}{2} - x\right)$$

refleksjon om linjen $y = x$.

$$(\cos x)' = \cos\left(\frac{\pi}{2} - x\right)\cdot \underbrace{\left(\frac{\pi}{2} - x\right)'}_{-1}$$
$$= -\sin(x)$$

Hvis enheten til vinkelen er grader: $\sin\left(\frac{\pi}{180^\circ}\cdot x\right)$

$$\left(\sin\left(\frac{\pi}{180^\circ}\cdot x\right)\right)' = \frac{\pi}{180^\circ}\cos\left(\frac{\pi}{180^\circ}\cdot x\right)$$

Hovedgrunnen til at vi benytter enheten radianer, er at den deriverte til sin x blir da enklest mulig.

(Hvis vi har en vinkelenhet slik at ett omløp svarer til 1

$$(\sin(2\pi \cdot x))' = 2\pi \cos(2\pi \cdot x)$$)

Produktregelen

$$(f \cdot g)(x) = f(x) \cdot g(x)$$

$$(f \cdot g)' = f' \cdot g + f \cdot g'$$

$$(\tan x)' = \left(\frac{\sin(x)}{\cos(x)}\right)' = \left(\sin(x) \cdot \frac{1}{\cos(x)}\right)'$$

$$= (\sin x)' \frac{1}{\cos x} + \sin x \underbrace{\left(\frac{1}{\cos x}\right)'}_{\frac{-1}{\cos^2 x}(\cos x)'}$$ kjerneregel

$$= \cos x \cdot \frac{1}{\cos x} + \sin x \left(\frac{-1}{\cos^2 x}\right) \cdot (-\sin x)$$

$$= 1 + \frac{\sin^2 x}{\cos^2 x} = \underline{1 + \tan^2(x)} = \frac{1}{\cos^2 x}$$

Øving

Oppg. 1

$$(3\sin x - 2\cos x)' = 3(\sin x)' + (-2)(\cos x)'$$

$$= \underline{3\cos x + 2\sin x}$$

Oppg. 2

$$(\sin(2x) + \sin^2 x)'$$ kjerneregel

$$= \cos(2x) \cdot (2x)' + 2\sin x \cdot (\sin x)'$$

$$= 2\cos(2x) + \underbrace{2\sin x \cos x}_{\sin 2x}$$

$$= \underline{2\cos(2x) + \sin(2x)}.$$

oppg 3

$$\cos x \cdot \tan x = \cos x \cdot \frac{\sin x}{\cos x} = \sin x$$

prod.regelen

$$(\cos x \cdot \tan x)' = (\cos x)' \cdot \tan x + \cos x \cdot (\tan x)'$$

$$= -\sin x \tan x + \cos x \,(1/\cos^2 x) = (1-\sin^2 x)/\cos x$$

$$= \cos^2 x / \cos x = \cos x.$$

oppg 4

$$2\sin x \cdot \cos x = \sin 2x$$

I

$$(2\sin x \cdot \cos x)' = 2\big(\underbrace{(\sin x)'}_{\cos x} \cdot \cos x + \sin x \underbrace{(\cos x)'}_{-\sin x}\big)$$

$$= 2(\cos^2 x - \sin^2 x)$$

$$= \underline{\underline{2\cos(2x)}}$$

II

$$(\sin 2x)' = \cos(2x) \cdot (2x)'$$

$$= \underline{2\cos(2x)}$$

oppg 5

$$2\cos^2 x - \cos(2x) = 2\cos^2 x - (\cos^2 x - \sin^2 x)$$

$$= \cos^2 x + \sin^2 x = 1 \text{ alle } x.$$

så den deriverte er 0

Alternativt

$$2 \cdot 2\cos(x) \cdot (\cos x)' - (-\sin 2x)(2x)'$$

$$-4\cos x \sin x + 2\sin 2x = -4\cos x \sin x + 2 \cdot 2 \sin x \cos x = 0.$$

oppg 6

$$(\cos^3(4x-1))' = 3\cos^2(4x-1) \cdot (\cos(4x-1))'$$

$$= 3\cos^2(4x-1)\left(-\sin(4x-1) \cdot \underbrace{(4x-1)'}_{4}\right) = -12\cos^2(4x-1)\sin(4x-1)$$

oppg 7

$$\left(x^2 \sin(-3x+4)\right)' = 2x\sin(-3x+4) + x^2(\sin(-3x+4))'$$

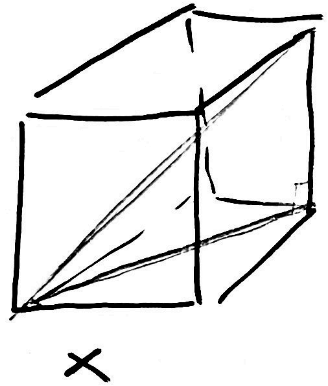
$$= 2x\sin(-3x+4) + x^2\sin(-3x+4) \cdot \underbrace{(-3x+4)'}_{-3}$$

$$= 2x\sin(-3x+4) - 3x^2\sin(-3x+4)$$

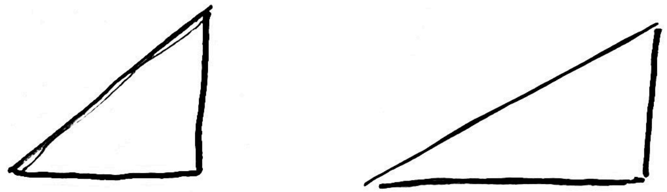
$$= \underline{2x\sin(-3x+4)}$$

oblig5 hint.

1 e)



Benytt Pytagoras sin sats to ganger



11e) $\sin(x) < \underbrace{\sin(2x)}_{2\sin x \cos x}$ trig. identiteter

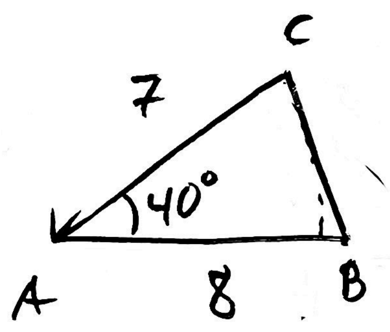
$\Leftrightarrow \quad 0 < \sin x\,(2\cos x - 1)$

Finn fortegn til $\sin x$

$2(\cos x - 1/2)$

2 ~~d~~)

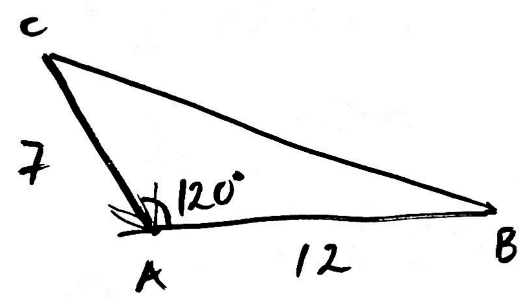
$\angle A = 40^\circ$ $AB = \sout{12}\ 8$ $AC = 7$



BC kan regnes ut ved å benytte cosinussetningen.

2f)

$\angle A = 120^\circ$ $AB = 12$ $AC = 7$



cosinussetning

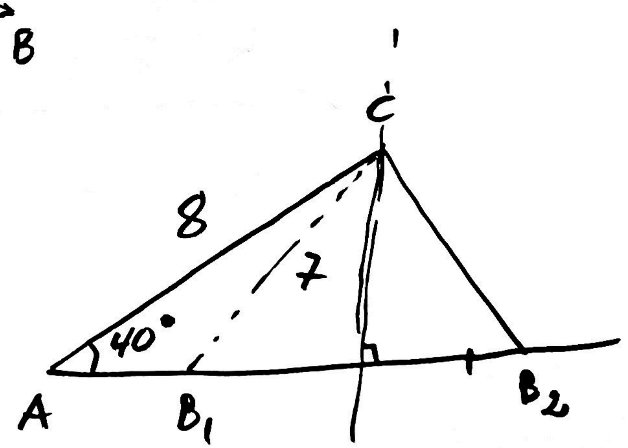
2e)

$\angle A = 40^\circ$ $AC = 8$ og $BC = 7$

$$\frac{\sin \angle A}{BC} = \frac{\sin(\angle B)}{AC}$$ så

$$\sin \angle B = \frac{8}{7} \cdot \sin(40^\circ)$$

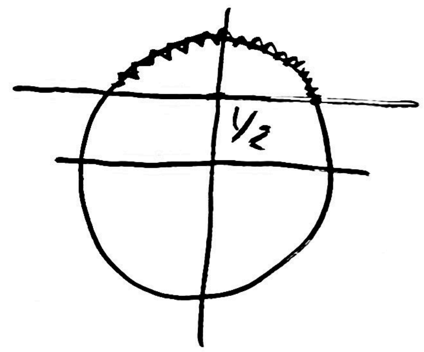
...



#3 Gjennomgått i en forelesning. Se der.

Eksempel

Løs $\sin(v) > 1/2 \qquad -\pi \le v \le \pi$



$\sin v = \frac{1}{2}$ når $v = \frac{\pi}{6} + 2\pi \cdot n$

$= \frac{5\pi}{6} + 2\pi \cdot n.$

Løsningene er $\underline{v \in \langle +\frac{\pi}{6}, \frac{5\pi}{6} \rangle}$.