27 januar 25

11. Trigonometriske funksjoner

A Trigonometriske likninger

Eksempel

$2\sin x = -1 \iff \sin x = \frac{-1}{2}$

$\cos x = 0.37$

$\tan x = -1$

$\sin x + \cos x = 0$

$(\iff \sin x = -\cos x \iff \tan x = -1)$

$\cos x$ må være ulik 0

$\sin^2 x - 3\sin x + 2 = 0$ 2.gradslikning i $\sin x$

$u^2 - 3u + 2 = 0$ $(u = \sin x)$

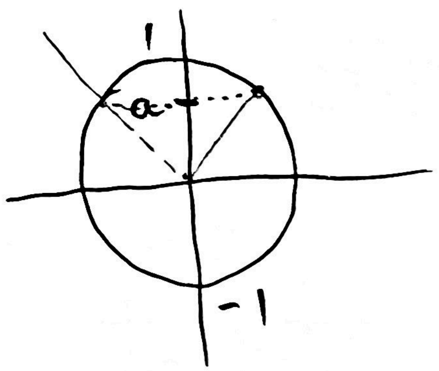
$(u-2)(u-1) = 0 \iff u = 1$ eller $u = 2$.

så løsningene er alle x slik at $\sin x = 1$ eller $\sin x = 2$

Sinuslikninger

$\sin(v) = a$ og $v \in [0°, 360°]$

en begrensning på løsningene.



Generelt er løsningene:

$|a| > 1$ ingen løsning

$a = 1$ $v = 90° + 360° \cdot n$ (hele omløp)

$a = -1$ $v = 270° + 360° \cdot n$

$|a| < 1$ $v = \arcsin(a) + 360° \cdot n$ (1 og 4 kvadrant)

$v = 180° - \arcsin(a) + 360° \cdot n$ (2 og 3 —)

(refleksjon om y-aksen)

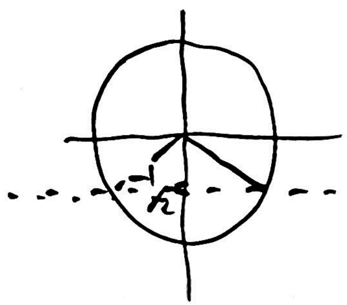
Hvis $v \in [0°, 360°]$, da får vi $a \geq 0$ $\arcsin a$, $180° - \arcsin a$

$a < 0$ $\quad v = 360° + \arcsin v$ $\quad$ (4. kvadrant)

$v = 180° - \arcsin v$ $\quad$ 3 kvadrant

$\sin v = -1/2$ $\quad v \in [0°, 360°]$

$(\quad [0, 2\pi])$



$v = \arcsin(-1/2) + 360° \cdot n$

og $v = 180° - \arcsin(-1/2) + 360° \cdot n$

$\Big(\arcsin(-1/2) = -\arcsin(1/2) = -30°\Big)$

$v = -30° + 360° \cdot n$

$v = 180° - (-30°) + 360° \cdot n$

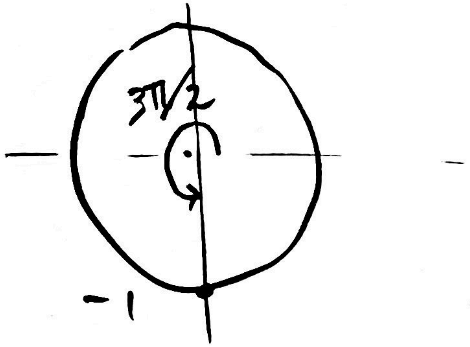
Løsningene er $\underline{v \in \{330°, 210°\}}$

Med radianer: $v = -\frac{\pi}{6} + 2\pi = \frac{-\pi}{6} + \frac{12\pi}{6} = \frac{11\pi}{6}$

og $v = \pi - \left(\frac{-\pi}{6}\right) = \frac{6\pi}{6} + \frac{\pi}{6} = \frac{7\pi}{6}$

blir løsningene $v \in \left\{\frac{7\pi}{6}, \frac{11\pi}{6}\right\}$

oppg Løs likningen $\sin v = -1$ $v \in [0, 4\pi]$



$$v = \underset{\substack{\uparrow \\ \text{én løsning}}}{\frac{3\pi}{2}} + \underset{\substack{\uparrow \\ \text{hele omløp.}}}{2\pi \cdot n}$$

løsninger for $n = 0$ og 1.

$$v \in \underline{\left\{ \frac{3\pi}{2}, \frac{7\pi}{2} \right\}}$$

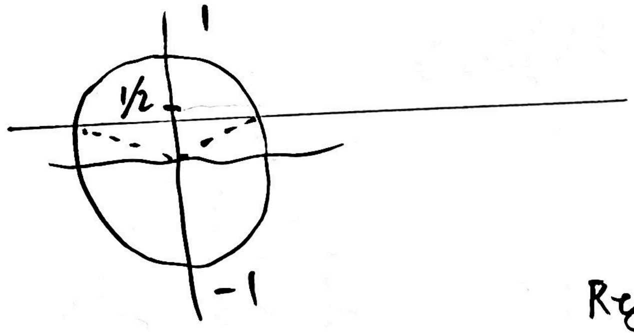
$$\left(2\pi + \frac{3\pi}{2} = \frac{4\pi + 3\pi}{2} \right)$$

alternativt: løsningene er $v = \frac{3\pi}{2}$ og $v = \frac{7\pi}{2}$

(Med enheten grader: $\underline{270°}$ og $360° + 270° = \underline{630°}$)

eks

$\sin v = 0.42 \qquad v \in [0, 390°]$.



én løsning

$v = \arcsin(0.42) = \sin^{-1}(0.42) \sim \underline{24.83°}$

Reflekterer om y-aksen

$v = 180° - \arcsin(0.42) \sim \underline{155.17°}$

$$v = \begin{cases} 24.83° + 360° \cdot n & n = 0, 1 \\ 155.17° + 360° \cdot n. & n = 0 \end{cases}$$

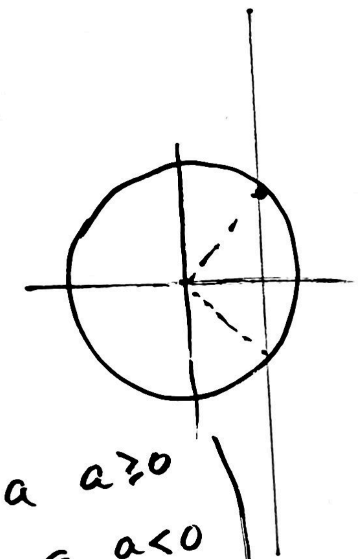
Løsningene er $\{ 24.83°, 155.17°, 384.83° \}$

(Eksakte verdier: $\{ \arcsin(0.42), 180° - \arcsin(0.42), \arcsin(0.42) + 360° \}$)

Cosinuslikninger

$\cos(V) = a$

begrensning på mulige løsninger



$|a| > 1$ ingen løsning

$a = 1 \quad V = 0° + 360° \cdot n$

$a = -1 \quad V = 180° + 360° \cdot n$

$|a| < 1 \quad V = \arccos(a) + 360° \cdot n$

og $V = -\arccos(a) + 360° \cdot n$

$\left(|a| = \begin{cases} a & a \geq 0 \\ -a & a < 0 \end{cases} \right.$

$|3| = 3$

$\left. |-4| = 4 \right)$

Eks

$\cos(V) = \frac{1}{\sqrt{2}}$ (figuren ovenfor)

$V \in [0, 2\pi]$.

$V_1 = \arccos\left(\frac{1}{\sqrt{2}}\right) + 2\pi \cdot n$

$= \frac{\pi}{4} + 2\pi \cdot n \quad n = 0$

$V_2 = -\frac{\pi}{4} + 2\pi \cdot n \quad n = 1$

Løsningene til $\cos(v) = 1/\sqrt{2}$

er $\underline{v = \frac{\pi}{4}}$ og $v = \frac{-\pi}{4} + 2\pi \cdot \overset{1}{n} = \frac{-\pi}{4} + \frac{8\pi}{4}$

$= \underline{\frac{7\pi}{4}}$

$$\underline{v \in \left\{\frac{\pi}{4}, \frac{7\pi}{4}\right\}}$$

$v \in [0, 360°]$

oppg

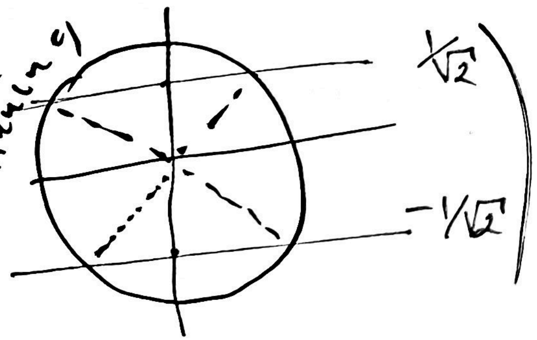
$\cos^2 v = \frac{1}{2}$

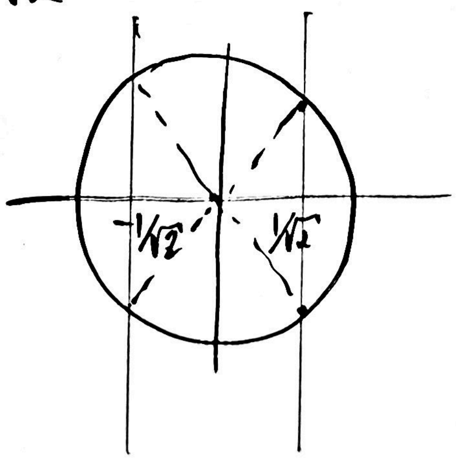
$\left((\cos v)^2 = \frac{1}{2} \right)$

$\left(u^2 = \frac{1}{2} \text{ har løsninge } u = \frac{\pm 1}{\sqrt{2}} \right)$

$\cos v = \frac{1}{\sqrt{2}}$ eller $\cos v = \frac{-1}{\sqrt{2}}$

(ikke en sinus-likning)





$\cos v = \frac{1}{\sqrt{2}}$ løst ovenfor

$v \in \{45°, 315°\}$

$\cos V = -\frac{1}{\sqrt{2}}$ $\qquad V = \underbrace{\pm \arccos\left(\frac{-1}{\sqrt{2}}\right)}_{\pm 135^\circ} + 360^\circ \cdot n$

$\pm 135^\circ + 360^\circ \cdot n$

$135^\circ + 360^\circ \cdot n$ mellom 0° og 360°

$-135^\circ + 360^\circ \cdot n$ er løsningene

$135^\circ, 225^\circ$

Løsningene til $\cos^2 V = \frac{1}{2}$ $\qquad V \in [0^\circ, 360^\circ]$

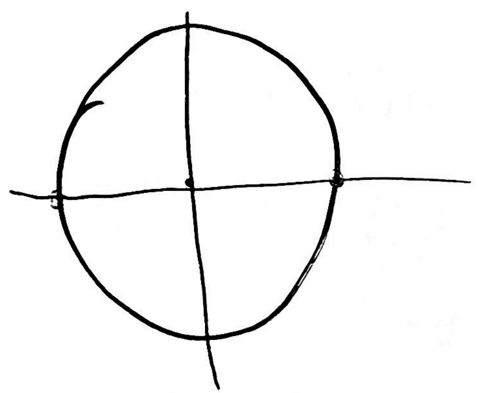
er $V \in \underline{\{45^\circ, 135^\circ, 225^\circ, 315^\circ\}}$

Eks

$\sin V \cos V = 0 \qquad V \in [0, 2\pi\rangle$

(Husk: $a \cdot b = 0 \iff a = 0$ eller $b = 0$)

$\sin V \cos V = 0 \iff \sin V = 0$ eller $\cos V = 0$.



$\sin V = 0$

$V = 0$ eller π radian

$\cos V = 0$

$V = \frac{\pi}{2}$ eller $\frac{3\pi}{2}$ radian.

Løsningene til $\sin V \cos V = 0$ er

$V \in \{0, \frac{\pi}{2}, \pi, 3\pi/2\}$

Alternativt: $\sin V \cos V = \frac{1}{2}\sin(2V)$

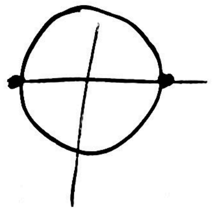
så foregående likning er ekvivalent til

$\frac{1}{2}\sin(2V) = 0 \iff \sin(2V) = 0.$

La $u = 2V$ 1) Løser $\sin(u) = 0$

2) Løser for V: $V = \frac{u}{2}.$

$\sin(u) = 0$



$u_1 = 0 + 2\pi \cdot n$

og $u_2 = \pi + 2\pi \cdot n$

$v = \frac{u}{2}$ så løsningene er

$v_1 = \pi \cdot n$

$v_2 = \frac{\pi}{2} + \pi \cdot n$

(i $[0, 2\pi\rangle$ får vi selvsagt de samme løsningene som i foregående eksempel)

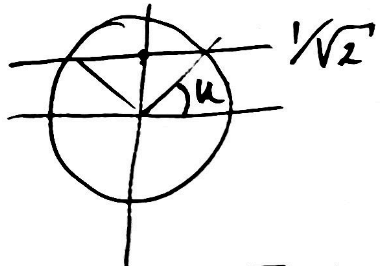
Eks.

$\sin v \cos v = \frac{1}{2\sqrt{2}}$.

$\Leftrightarrow \underbrace{2 \sin v \cos v}_{\sin(\underbrace{2v}_{u})} = \frac{1}{\sqrt{2}}$

$\sin(\underbrace{2v}_{u}) = 1/\sqrt{2}$.

$2v = u$ så $v = \frac{u}{2}$



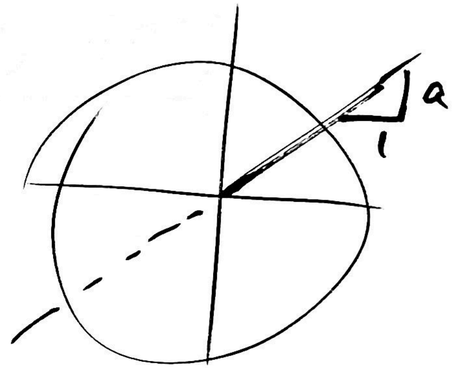
$u_1 = \frac{\pi}{4} + 2\pi \cdot n$

$u_2 = \frac{3\pi}{4} + 2\pi \cdot n$

$$\underline{V_1 = \frac{u_1}{2} = \frac{\pi}{8} + \pi \cdot n}$$

$$\underline{V_2 = \frac{u_2}{2} = \frac{3\pi}{8} + \pi \cdot n}$$

$$\tan x = a$$

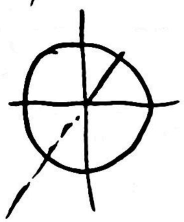


$$x = \arctan(a) + \pi \cdot n$$

$$n \in \mathbb{Z}.$$

Øving

oppg 1) $\tan V = \sqrt{3}$ $\qquad V \in [-180°, 180°]$

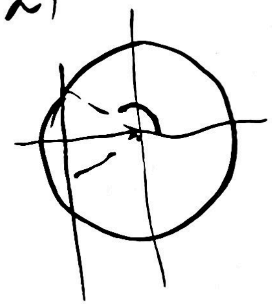


$V = \arctan \sqrt{3} = 60°$

Alle løsningene er $V = 60° + 180° \cdot n$

Løsningene er $V \in \underline{\{-120°, 60°\}}$

2) $\cos(V) = -\sqrt{3}/2$ $\qquad V \in [0, 2\pi]$



én løsning: $V = \arccos(-\sqrt{3}/2) = 150°$

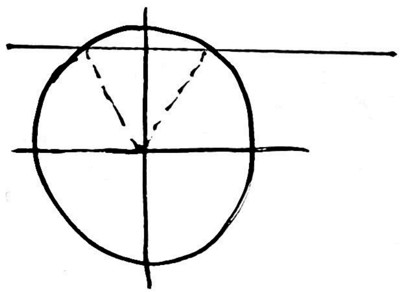
$= \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

$V_1 = \frac{5\pi}{6} + 2\pi \cdot n \qquad n = 0$

$V_2 = \frac{-5\pi}{6} + 2\pi \cdot n \qquad n = 1$

Løsningene er $v \in \underline{\{\frac{5\pi}{6}, \frac{7\pi}{6}\}}$

3) $\sin(V) = \frac{\sqrt{3}}{2}$ $\qquad V \in [-\pi, 5\pi]$



én løsning $\qquad V_1 = \arcsin\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$.

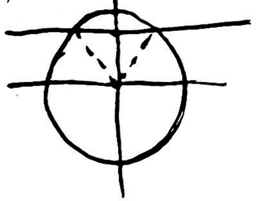
Reflekterer om y-aksen $\qquad \pi - V_1 = \frac{2\pi}{3}$

Alle mulige løsninger $\qquad V_1 = \frac{\pi}{3} + 2\pi \cdot n \qquad n = 0, 1, 2$

$V_2 = \frac{2\pi}{3} + 2\pi \cdot n$

Løsningene er $V = \left\{ \underline{\frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}, \frac{13\pi}{3}, \frac{14\pi}{3}} \right\}$

4) $\sin V = 0.75 = 3/4$ $\qquad V \in [0, 360°]$



én løsning $\qquad \arcsin(3/4) \sim 48.59°$

den andre løsningen : $\qquad 180° - \arcsin(3/4)$

$\sim \underline{131.41}°$

Løsningene er $V \in \{48{,}59°, 131.41°\}$

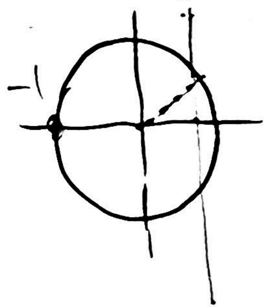
5) $4\cos^2 v + \cos v - 3 = 0$ 2.gradslikning i $u = \cos v$.

$u^2 + \frac{1}{4}u - \frac{3}{4} = 0$

$(u+1)(u-\frac{3}{4}) = 0 \Leftrightarrow u+1=0$ eller $u-\frac{3}{4}=0$

$\Leftrightarrow u = -1$ ella $u = \frac{3}{4}$.

$\Leftrightarrow \cos v = -1$ eller $\cos v = 3/4$



$\cos v = -1$: $\underline{v = 180° + 360° \cdot n}$

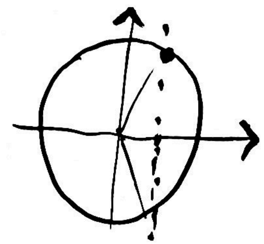
$\cos v = 3/4$: $v_1 = \arccos(3/4) + 360° n \,\hat{=}\, \underline{41.41° + 360° \cdot n}$

og $v_2 = -\arccos(3/4) + 360° \cdot n \approx \underline{-41.41° + 360° \cdot n}$

6) $\cos(x+1) = 1/2$ $\qquad x \in [0, 6]$

$V = x+1$

$\cos(V) = 1/2$



$$V_1 = \arccos\left(\tfrac{1}{2}\right) + 2\pi \cdot n$$
$$= \frac{\pi}{3} + 2\pi \cdot n$$
$$V_2 = \frac{-\pi}{3} + 2\pi \cdot n \ .$$

$\pi/3 \sim 1.047\ldots$

$2\pi \sim 6.283\ldots$

$x = V - 1$

$$x_1 = \frac{\pi}{3} - 1 + 2\pi \cdot n$$
$$x_2 = \frac{-\pi}{3} - 1 + 2\pi \cdot n$$

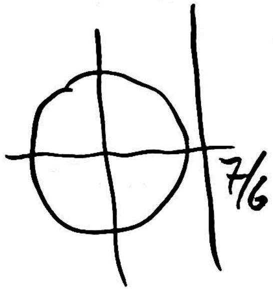
$x = 0.047\ldots$, $x = \frac{-\pi}{3} - 1 + 2\pi \sim 4.235\ldots$

Løsningene er $x \in \underline{\{0{,}047\ldots,\ 4.235\ldots\}}$

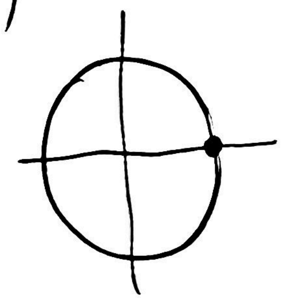
7) $3\cos(v) - 2 = 3/2$ $\qquad v \in [0, 360°]$

$3\cos(v) = 2 + \cancel{3/2} = 3.5 = 7/2$

$\cos(v) = 7/6 > 1$ $\qquad$ ingen løsning.



8) $\cos(v) = 1$ $\qquad v \in [0, 2\pi]$

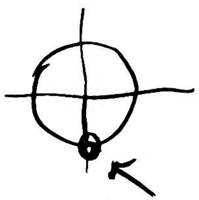


$v = 0$

$v = 2\pi \quad (= 360°)$

9) $\sin^3(v) = -1$ $\qquad v \in [0, 2\pi]$.

$((\sin(v))^3 = -1) \iff \sin(v) = -1$



$\underline{v = \frac{3\pi}{2}}$