9E Optimering

8 jan 25

Eksempel $\quad f(x) = x + \frac{1}{x}$ $\quad x > 0$

$D_f = \langle 0, \infty \rangle$

Når er $f(x)$ minst mulig?

$f(1) = 2$ $\quad f(2) = 2 + \frac{1}{2}$ $\quad f\left(\frac{1}{2}\right) = \frac{1}{2} + \frac{1}{1/2} = 2 + \frac{1}{2}$

(Vi har: $f(x) = f\left(\frac{1}{x}\right)$)

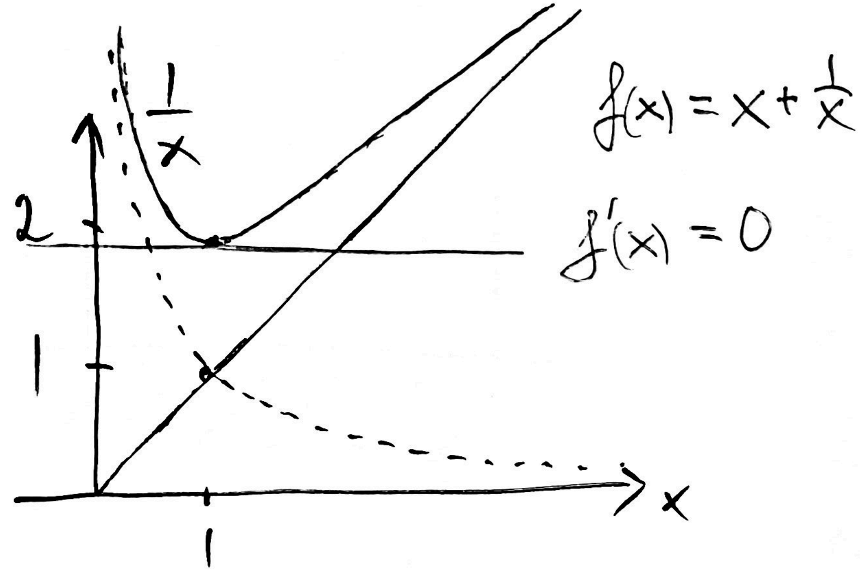
$$f(x) - 2 = x + \frac{1}{x} - 2 = \frac{1}{x}\left(x^2 + 1 - 2x\right)$$

$$= \frac{1}{x}(x-1)^2 \geq 0$$

og lik 0 når $x = 1$.

Vi har vist at $f(x) \geq 2$ for alle x

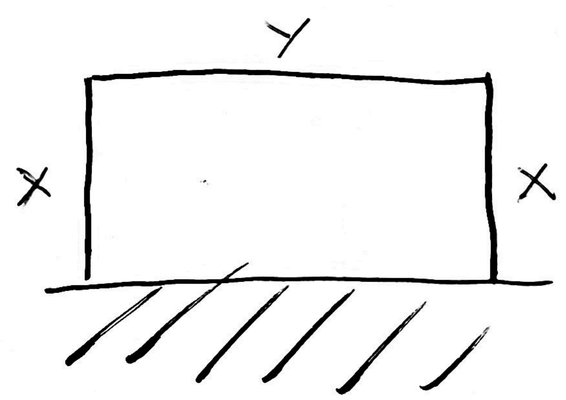
$f(x) = 2$ når $x = 1$



$$f'(x) = \left(x + \frac{1}{x}\right)'$$
$$= (x)' + \left(x^{-1}\right)'$$
$$= 1 + (-1) \cdot x^{-1-1}$$
$$f'(x) = 1 - \frac{1}{x^2}.$$

$$f'(x) = 0 = 1 - \frac{1}{x^2}$$
$$\Leftrightarrow 1 = \frac{1}{x^2} \Leftrightarrow x^2 = 1 \Leftrightarrow x = 1 \text{ og } x = -1.$$

Grafen er minst når $x = 1$
til funksjonen



Gjerde 30meter langt.
Det skal settes opp inntil en lang vegg.
Hvordan skal det gjøres for at innhegningen skal bli størst mulig (størst areal)?

$2x+y=30$ $\qquad$ $y = 30 - 2x$

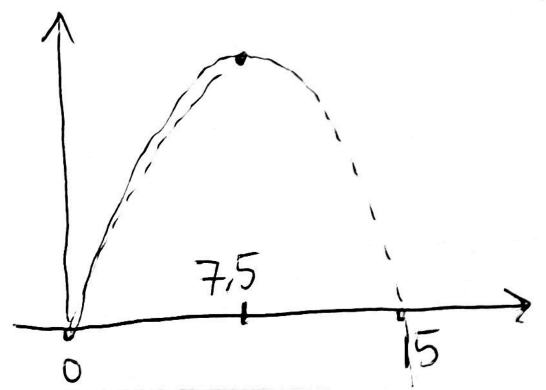
Areal $\quad A = x \cdot y$

$A(x) = x(30-2x)$ parabel

$A'(x) = (30x - 2x^2)' = 30(x)' - 2(x^2)'$

$= 30 - 4x = 0$

$\Leftrightarrow 4x = 30 \Leftrightarrow x = \frac{30}{4} = \underline{7.5 \text{ meter}}$



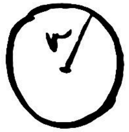
$y = 15$ meter.

Areal $x \cdot y = 7.5 \cdot 15 = 75 + 37.5$

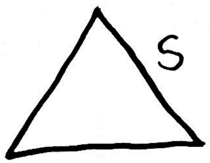
$= \underline{\underline{112.5 \text{ m}^2}}$

$$\overbrace{x \qquad 1-x}^{1}$$

$\downarrow$ Lager en sirkel



$\downarrow$ Lager en likesidet trekant



Når er summen av arealene minst mulig?

Sirkel:

Omkrets $O = 2\pi r = x$

Arealet $A_s = \pi r^2$

$= \pi \left(\frac{x}{2\pi}\right)^2$

$= \frac{\pi}{4\pi^2} x^2$

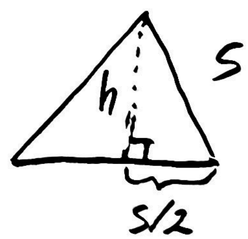
$$\underline{A_s(x) = \frac{1}{4\pi} \cdot x^2.}$$

Likesida trekant

$O = s+s+s = 3s$, $s = \frac{1-x}{3}$.

$A_t = \frac{1}{2} s \cdot h = \frac{\sqrt{3}}{4} s^2$

$= \frac{\sqrt{3}}{4} \frac{(1-x)^2}{3^2}$. $= \frac{\sqrt{3}}{4} \cdot \frac{(1-x)^2}{9}$



$h^2 + \left(\frac{s}{2}\right)^2 = s^2$

$h^2 = s^2 - \frac{s^2}{4}$

$= \frac{3}{4} s^2$

$$\underline{h = \frac{\sqrt{3}}{2} s}$$

Totalt areal $A = A_s + A_t$

$$A = \frac{x^2}{4\pi} + \frac{\sqrt{3}}{4 \cdot 9} (1-x)^2$$

$$A'(x) = \frac{1}{4\pi} \cdot 2x + \frac{\sqrt{3}}{4 \cdot 9}\, 2(1-x) \underbrace{(1-x)'}_{-1}$$

$$= \frac{1}{2}\left[\frac{1}{\pi} \cdot x + \frac{\sqrt{3}}{9}(x-1) \right] = 0$$

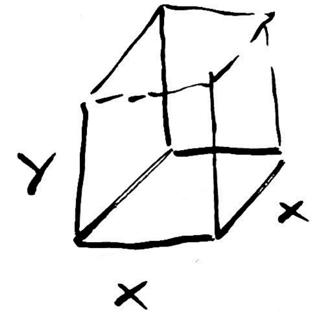
$$\frac{1}{\pi} x + \frac{\sqrt{3}}{9} x = \frac{\sqrt{3}}{9}$$

$$\left(\frac{1}{\pi} + \frac{\sqrt{3}}{9}\right) x = \frac{\sqrt{3}}{9}$$

$$x = \frac{\sqrt{3}/9}{1/\pi + \sqrt{3}/9} \cdot \frac{9}{9} \cdot \frac{\pi}{\pi}$$

$$\underline{x = \frac{\sqrt{3}\,\pi}{9 + \sqrt{3} \cdot \pi}} \sim 0.37\underline{6}8$$

Rett prisme



Volum $V = x^2 y$ fast.

Når er overflaten minst mulig?

1 Lukket boks $\quad a = 2$

2. Åpen boks med bunn. $\quad a = 1$

3. Uten topp og bunn. $\quad a = 0$

$$O = 4x \cdot y + ax^2$$

$$V = x^2 y$$
$$y = V/x^2$$

$$O(x) = 4x \cdot \frac{V}{x^2} + ax^2$$
$$= 4V \cdot \frac{1}{x} + ax^2.$$

$$O'(x) = 4V(x^{-1})' + a(x^2)' = \frac{-4V}{x^2} + 2a \cdot x$$

$$O'(x) = 0 \; : \; +\frac{4V}{x^2} = 2ax \Leftrightarrow \frac{4V}{2a} = x^3, \; a \neq 0$$

$$x = \sqrt[3]{\frac{2V}{a}}$$

Lukket boks $a=2$: $x = \sqrt[3]{V}$ $y = \frac{V}{(\sqrt[3]{V})^2} = \frac{V^1}{(V^{1/3})^2}$

$= V^1 \cdot V^{-2/3} = V^{1/3}$.

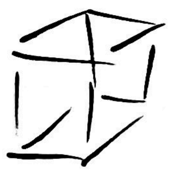
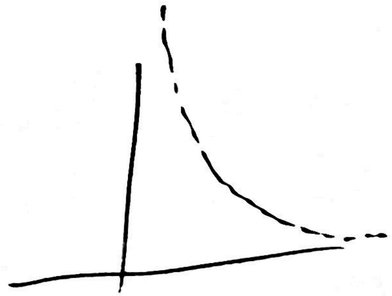
Så $x = y$

Åpen boks: $a=1$ $x = \sqrt[3]{2V}$ $y = V^1 (2V)^{-2/3}$

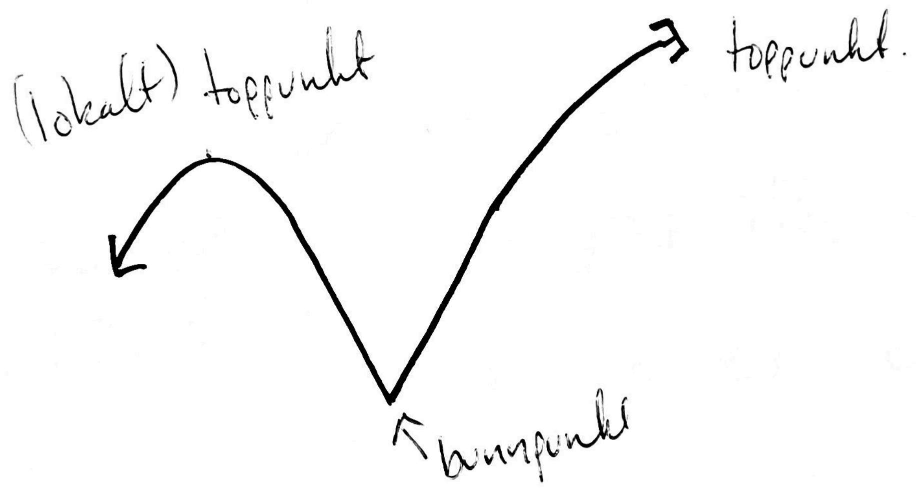
$= \frac{1}{2^{2/3}} \cdot \sqrt[3]{V}$

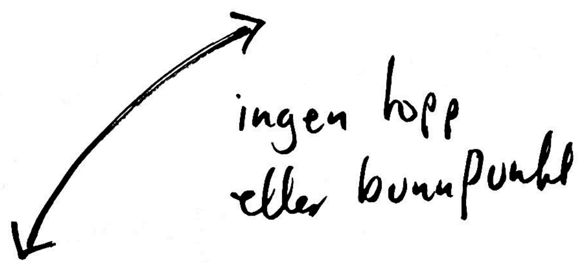
$$\frac{y}{x} = \frac{(1/2^{2/3}) \cdot \sqrt[3]{V}}{\sqrt[3]{2} \quad \sqrt[3]{V}}$$

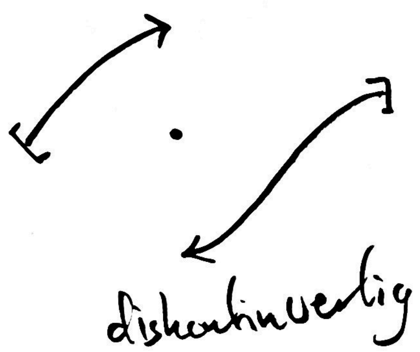
$$= \frac{1}{2^{2/3} \cdot 2^{1/3}} = \underline{\frac{1}{2}}.$$

$a=0$  $O = \frac{4V}{x}$ 

ingen optimal verdi for x. Overflaten blir mindre når x øker.





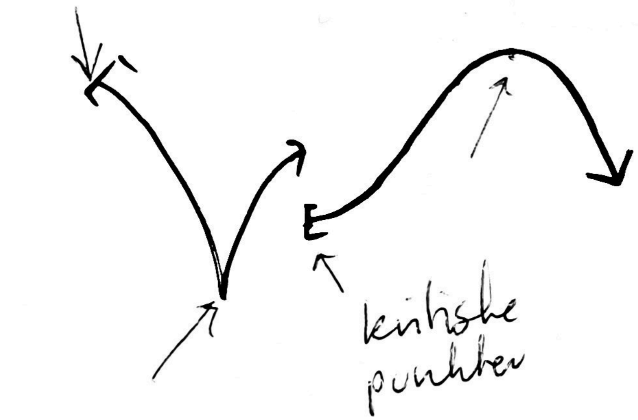


ingen globale ekstremalverdi

Ekstremalverdisetningen : Hvis $f(x)$ er kontinuerlig på en lukket og begrenset intervall $[a, b]$, så har $f(x)$ et globalt topp og bunnpunkt.

Kritiske punkter er verdier x slik at

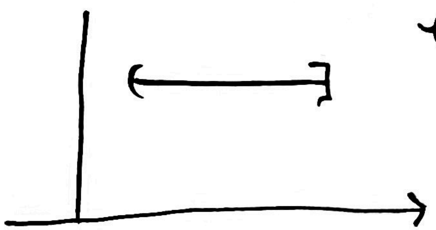
* $f'(x) = 0$
* $f(x)$ ikke er derivarbar i punkt
* x er et endepunkt.



$f(x)$ har ekstremalverdier blant de kritiske punktene til $f(x)$

Det er derfor tilstrekkelig å lete blant disse.

Alle punkt på grafen er både topp og bunnpunkt.

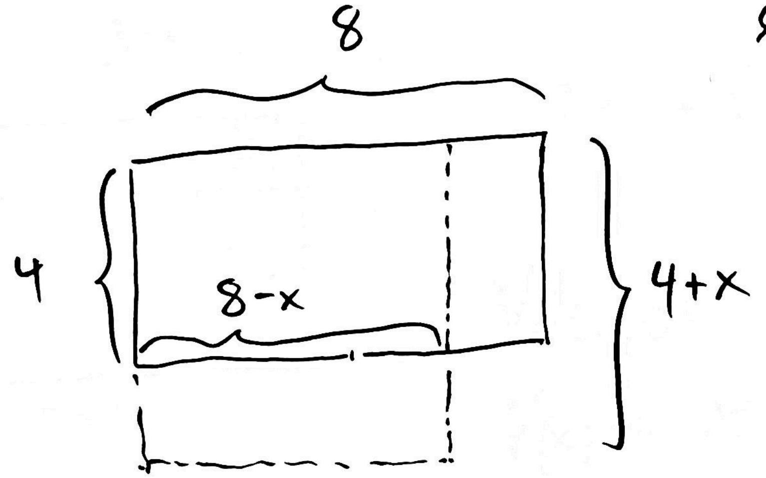


ØVING

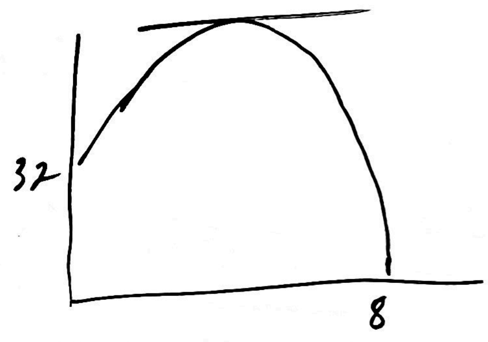
Oppg 9.48

$0 \leq x \leq 8$

a)



b) $A = (8-x)(4+x) = 32 + 4x - x^2.$



c) Når er A størst?

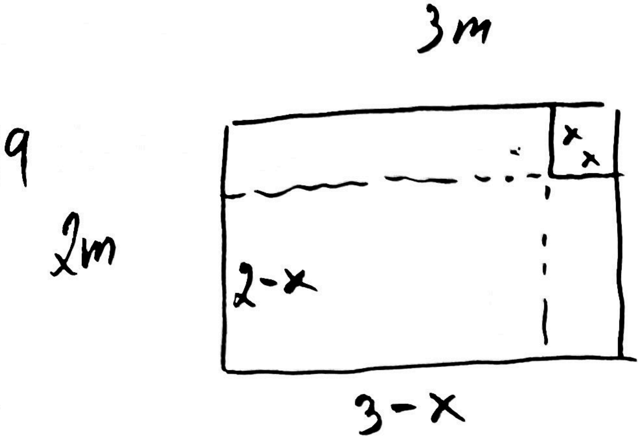
$A'(x) = 0 = -2x + 4 + 0 = 0$

$2x = 4$ så $\underline{x = 2}$.

$A(2) = (8-2)(4+2) = 6^2 = \underline{36}$

Arealet er størst når $x=2$. Arealet er da lik $36\,dm^2$

9.49



$D_V = [0, 2]$

a)

Volumet

$V(x) = x(2-x)(3-x)$

$= x(6-5x+x^2)$

$= x^3 - 5x^2 + 6x$

3.grads uttrykk.

b) $V'(x) = 3x^2 - 5(2x) + 6 = 3x^2 - 10x + 6 = 0$

$x = \frac{10 \pm \sqrt{10^2 - 4\cdot 3\cdot 6}}{2\cdot 3} = \frac{10 \pm \sqrt{100-72}}{6}$

$= \frac{10 \pm \sqrt{28}}{6}$ mellom 0 og 2 får vi løsninge-

$x = \frac{10 - \sqrt{4.7}}{6} = \frac{2(5-\sqrt{7})}{2\cdot 3} = \frac{5-\sqrt{7}}{3} \approx 0.785m$

$\approx 78.5cm$

Setter inn denne verdien i uttrykket for Volumet og får at største volum er $2.11m^2$

Fortjeneste = antall enheter solgt · ~~pris~~ fortjeneste/enhet

$$F(p) = 10^5 \cdot 2 \left(\tfrac{1}{2}\right)^{\frac{p}{100}} (p - 100)$$

(# solgte enheter halveres når p økes med 100)

$F(100) = 0$

enheter solgt ved $p = 100$ er 10^5

$$F(p) = (2 \cdot 10^5)\, 2^{-p/100} \left(\frac{p}{100} - 1\right) \cdot 100$$

La $x = \frac{p}{100}$ $\qquad F = 2 \cdot 10^5\, 2^{-x} (x-1) \cdot 100$

$$\frac{d}{dx} F = 2 \cdot 10^{5+2} \left((2^{-x})'(x-1) + 2^{-x}(x-1)' \right)$$

$$= 2 \cdot 10^7 \left((-\ln 2)\, 2^{-x}(x-1) + 2^{-x} \cdot 1 \right) = 0$$

$$-\ln 2 (x-1) + 1 = 0$$

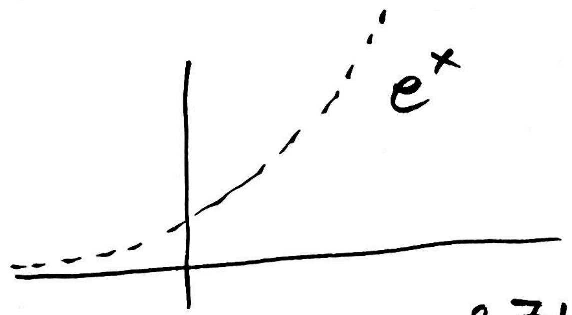
$$\ln 2 \cdot x - \ln 2 = 1$$

$$x = \frac{1 + \ln 2}{\ln 2}$$

$p = 100x = 100 \frac{1+\ln 2}{\ln 2} = \underline{244.3}$ gir optimal fortjeneste.

Forklarer hvorfor

$$(e^x)' = e^x$$



$$e \approx 2.71828\ldots$$

$$a = e^{\ln a}$$

$$a^x = (e^{\ln a})^x$$

$$= e^{\ln a \cdot x}$$

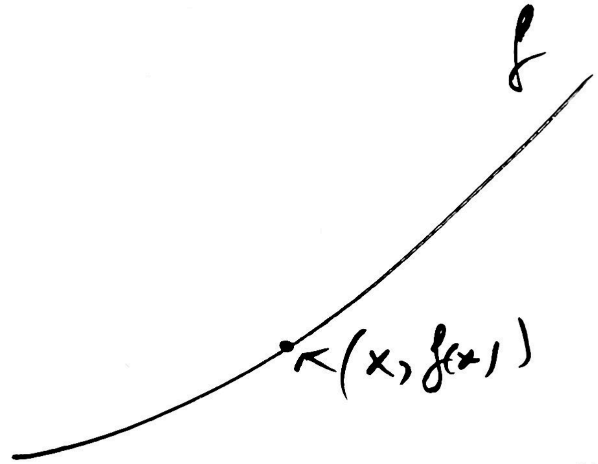
$$(a^x)' = \left(e^{(\ln a \cdot x)}\right)' = e^{\ln a \cdot x} \cdot \overbrace{(\ln a \cdot x)'}^{\ln a}$$

kjerneregelen

$$\underline{(a^x)' = \ln a \cdot a^x}$$

$$\left(a = \tfrac{1}{2}, \quad \ln\left(\tfrac{1}{2}\right) = \ln(2^{-1}) = -1 \cdot \ln(2) = \underline{-\ln(2)}. \right)$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$



$$f(x) = a^x$$

$$(a^x)' = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h}$$

$$= \lim_{h \to 0} \frac{a^x(a^h - 1)}{h}$$

$$= a^x \underbrace{\lim_{h \to 0} \frac{a^h - 1}{h}}_{\text{Uavhengig av } x.}$$

$$a^{x+h} = a^x \cdot a^h$$

Visualisert i GeoGebra
$h = 10^{-n}$
$0 \leq n \leq 10$
$1 \leq a \leq 5$
Benyttet glidere for n og a.

Det er et tall a
$2.718 < a < 2.719$
slik at grensen blir lik 1.

Euler tallet e er tallet s.a. $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$

Vi får da
$(e^x)' = 1 \cdot e^x = e^x$.