17.4.23

Delbrøksoppspalting

17.7

Integrasjon av rasjonale funksjoner.

$$\int \frac{1}{x}\,dx = \ln|x| + C \qquad \int x^r\,dx = \frac{x^{r+1}}{r+1} + C, \quad r \neq -1$$

$$\int \frac{1}{-3x+5}\,dx \qquad \left(\begin{array}{l} u = -3x+5 \\ u' = -3 \\ u'dx = du \\ -3\,dx = du \end{array} \right)$$

$$= \int \frac{1}{u}\left(\frac{1}{-3}\right)du$$

$$= \frac{-1}{3}\ln|u| + C$$

$$= \underline{\frac{-1}{3}\ln|-3x+5| + C}$$

$$\int \frac{x^2}{x+2}\,dx = \int x-2+\frac{4}{x+2}\,dx$$
$$= \underline{\frac{x^2}{2} - 2x + 4\ln|x+2| + C}$$

$$\int \frac{x^2}{(x+2)^2}\,dx$$

Polynomdivisjon

$$x^2 \quad - \quad - \; : \; x^2+4x+4 = 1 - 4\,\frac{x+1}{(x+2)^2}$$
$$\underline{x^2+4x+4}$$
$$-4x-4$$

$$\int 1 - 4\,\frac{x+1}{(x+2)^2}\,dx = \int 1 - 4\,\frac{x+2-1}{(x+2)^2}\,dx$$
$$= \int 1 - 4\left(\frac{1}{x+2} - \frac{1}{(x+2)^2}\right)dx = \int 1 + \frac{-4}{x+2} + \frac{4}{(x+2)^2}\,dx$$
$$= \underline{x - 4\ln|x+2| + \frac{-4}{(x+2)} + C}$$

$$\frac{1}{6} = \frac{1}{2\cdot 3} = \frac{1}{2} - \frac{1}{3}$$

$$\frac{1}{14} = \frac{1}{2\cdot 7} = \frac{7-3\cdot 2}{2\cdot 7} = \frac{1}{2} - \frac{3}{7}$$

Tilsvarende for polynomer: $\dfrac{r(x)}{p(x)\cdot q(x)} = \dfrac{s(x)}{p(x)} + \dfrac{t(x)}{q(x)}$

$\deg s < \deg p$
$\deg t < \deg q$

$\deg r < \deg p + \deg q$

<u>Delbrøksoppspalting</u>

p, q relativt primiske.

$$\frac{1}{(x+2)(x-3)} = \frac{A}{x+2} + \frac{B}{x-3}$$

A, B konstanter
(pol. av grad < 1)

Gange opp med $(x+2)(x-3)$:

$1 = A(x-3) + B(x+2)$ like for alle $x \neq -2, 3$.

Derfor identisk like, og derfor like for alle x

I

setter $x=3$: $1 = B \cdot 5$ så $B = 1/5$

$x=-2$: $1 = -5A$ så $A = -1/5$

II

$0x + 1 = (A+B)x + (2B-3A)$

like: $A + B = 0$ likningssystem

$2B - 3A = 1$

$A = -B$ så $2B - 3(-B) = 1$

$5B = 1$

så $B = \frac{1}{5}$, $A = -B = \frac{-1}{5}$.

$$\frac{1}{(x+2)(x-3)} = \frac{1}{5}\left(\frac{1}{x-3} - \frac{1}{x+2}\right) \checkmark$$

$$\int \frac{1}{x^2-x-6}\,dx \underset{\text{faktorisere}}{=} \int \frac{1}{(x+2)(x-3)}\,dx \underset{\text{delbrøks-oppspalting}}{=} \frac{1}{5}\int \frac{1}{x-3} - \frac{1}{x+2}\,dx$$

$$= \frac{1}{5}\left(\ln|x-3| - \ln|x+2|\right) + C$$

$$= \underline{\frac{1}{5}\ln\left|\frac{x-3}{x+2}\right| + C}$$

$$\frac{2x}{x^2-x-6} = \frac{2x}{(x-3)(x+2)} = \frac{A}{x+2} + \frac{B}{x-3}$$

$$2x = A(x-3) + B(x+2) \quad \text{for alle } x$$

$x=3$: $\quad 6 = 2\cdot 3 = 0 + 5B$ så $B = 6/5$

$x=-2$: $\quad -4 = -5A + 0$ så $A = 4/5$

så $$\int \frac{2x}{(x-3)(x+2)}\,dx = \int \frac{1}{5}\left(\frac{4}{x+2} + \frac{6}{x-3}\right)dx$$

$$= \frac{1}{5}\left(4\ln|x+2| + 6\ln|x-3|\right) + c$$

$$= \frac{2}{5}\left(2\ln|x+2| + 3\ln|x-3|\right) + c$$

$$= \underline{\frac{2}{5}\ln\left|(x+2)^2(x-3)^3\right| + C}$$

opg. $\displaystyle\int \frac{3}{x(x+1)}\,dx = \int \frac{3}{x^2+x}\,dx$

Delbrøksoppspalting

$$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$

$$1 = A(x+1) + Bx$$

$$0\cdot x + 1 = (A+B)x + A$$

$A = 1$ og $A+B=0$, så $B = -A = -1$

$$3\int \frac{1}{x(x+1)}\,dx = 3\int \frac{1}{x} - \frac{1}{x+1}\,dx = 3\left(\ln|x| - \ln|x+1|\right) + c$$

$$= \underline{3\ln\left|\frac{x}{x+1}\right| + C}$$

Alle rasjone funksjoner har en elementær antiderivert.

$$\int \frac{x}{x^2-5}\,dx$$

Faktoriserer x^2-5

$$x^2-5 = x^2 - (\sqrt{5})^2$$
$$= (x+\sqrt{5})(x-\sqrt{5})$$

$$\frac{x}{x^2-5} = \frac{A}{x+\sqrt{5}} + \frac{B}{x-\sqrt{5}}$$

$$x = A(x-\sqrt{5}) + B(x+\sqrt{5})$$

$x=-\sqrt{5}$: $\quad -\sqrt{5} = A \cdot 2(-\sqrt{5})$, $\quad A = 1/2$

$x=\sqrt{5}$: $\quad \sqrt{5} = B(2\sqrt{5})$ $\quad B = 1/2$

$$\int \frac{x}{x^2-5}\,dx = \frac{1}{2}\int \frac{1}{x+\sqrt{5}} + \frac{1}{x-\sqrt{5}}\,dx$$
$$= \frac{1}{2}\left(\ln|x+\sqrt{5}| + \ln|x-\sqrt{5}|\right) + C$$
$$= \frac{1}{2}\ln|(x+\sqrt{5})(x-\sqrt{5})| + C$$

$$\int \frac{x}{x^2-5}\,dx = \underline{\frac{1}{2}\ln|x^2-5| + C}$$

Dette kan gjøres enklere. Vi kan benytte substitusjon.

$$u = x^2-5 \qquad u' = 2x$$

$$du = 2x\,dx$$

$$\frac{1}{2}du = x\,dx$$

$$\int \frac{1}{x^2-5}\,x\,dx = \int \frac{1}{u}\,\frac{1}{2}du = \frac{1}{2}\ln|u| + C$$

$$= \underline{\frac{1}{2}\ln|x^2-5| + C}$$

$$\int \frac{2}{x(x-3)^2}\,dx = \int \frac{2}{x^3-6x^2+9x}\,dx$$

$$\frac{2}{x(x-3)^2} = \frac{A}{x} + \frac{Bx+C}{(x-3)^2}$$

$$* \quad 2 = A(x-3)^2 + (Bx+C)x$$

$x=3$: $\quad 2 = 3(3B+C) = 9B + 3C$

$x=0$: $\quad 2 = A(-3)^2 = 9A \qquad A = \underline{2/9}$

Deriveren $*$: $\quad 0 = 2A(x-3) + 2Bx + C$

Setter $x=3$: $\quad 2 \cdot 3 \cdot B + C = 0$

$$2 = 9B + 3C$$
$$6B + C = 0, \quad C = -6B$$

Så $\quad 2 = 9B + 3(-6B) = -9B$

$$B = -2/9, \quad C = 4/3 = 12/9$$

$$\frac{2}{x(x-3)^2} = \frac{2}{9} \cdot \frac{1}{x} + \frac{2}{9} \frac{(-x+6)}{(x-3)^2}$$
$$= \frac{2}{9} \cdot \frac{1}{x} + \frac{2}{9}\left(\frac{-(x-3)+3}{(x-3)^2} \right)$$

$$\int \frac{2}{x(x-3)^2}\,dx = \frac{2}{9}\int \frac{1}{x}\,dx + \frac{2}{9}\int \frac{-1}{x-3} + \frac{3}{(x-3)^2}\,dx$$

$$= \frac{2}{9}\left(\ln|x| - \ln|x-3| + 3\frac{(x-3)^{-1}}{-1}\right) + c$$

$$= \underline{\frac{2}{9}\left(\ln\left|\frac{x}{x-3}\right| - \frac{3}{x-3}\right) + c}$$

Øving (ikkje pensum)

$$\int \frac{1}{x^2+1}\,dx = \arctan(x) + C$$

$$(\arctan(x))' = \frac{1}{x^2+1}.$$

$$y = \tan x \qquad \frac{dy}{dx} = (\tan x)' = 1 + \tan^2 x$$

$$= 1 + y^2$$

$$x = \arctan(y)$$

$$\frac{dx}{dy} = \left(\frac{dy}{dx}\right)^{-1} \text{ ved kjerneregeln.}$$

$$\left(\begin{array}{c} x(y(x)) = x \\ \text{Deriverer m.h.t } x \\ \frac{dx}{dy} \cdot \frac{dy}{dx} = \frac{d}{dx}x = 1 \end{array} \right)$$

$$\frac{dx}{dy} = \frac{1}{1+y^2}. \quad \checkmark$$

$$\int \frac{1}{x^2+4}\,dx = \int \frac{1/4}{x^2/4+1}\,dx \qquad u = \frac{x}{2}$$

$$u' = \frac{1}{2}$$

$$= \int \frac{1/2}{u^2+1}\,du \qquad du = \frac{1}{2}dx$$

$$= \frac{1}{2}\arctan(u) + c$$

$$= \frac{1}{2}\underline{\arctan\left(\frac{x}{2}\right)} + c$$

$$\int \frac{1}{x^2+4x+5}\,dx = \int \frac{1}{(x+2)^2-4+5}\,dx = \int \frac{1}{(x+2)^2+1}\,dx$$

$$v = x+2$$

$$dv = dx$$

$$= \int \frac{1}{v^2+1}\,dv$$

$$= \arctan(v) + c = \underline{\arctan(x+2) + c}$$

$$\int \frac{1}{x^2-2x+10}\,dx$$

$$= \int \frac{1}{(x-1)^2-1+10}\,dx = \int \frac{1}{(x-1)^2+9}\,dx$$

$u = \frac{x-1}{3}$, $u^2 = \frac{(x-1)^2}{9}$

$3\,du = dx$

$$= \int \frac{1/9}{(x-1)^2/9+1}\,dx$$

$$= \frac{1}{9}\int \frac{1}{u^2+1}\,3\,du$$

$$= \underline{\frac{1}{3}\arctan\left(\frac{x-1}{3}\right)+C}$$

$$\int \frac{x+2}{x^2+1}\,dx = \int \frac{x}{x^2+1}\,dx + \int \frac{2}{x^2+1}\,dx$$

$u = x^2+1$

$u' = 2x$

so $x\,dx = \frac{1}{2}du$

$$\int \frac{x+2}{x^2+1}\,dx = \frac{1}{2}\ln|x^2+1| + 2\arctan(x) + C$$

$u = x+2$, $x = u-2$

$x^2+4x+5 = u^2+1$

$$\int \frac{x}{x^2+4x+5}\,dx$$

$$= \int \frac{u-2}{u^2+1}\,du = \frac{1}{2}\ln(x^2+4x+5) - 2\arctan(x+2) + C$$

Alle reelle pol. er et produkt a lineære pol. og kvadratiske polynome (>0).

og $$\int \frac{1}{x^4-1}\,dx = \int \frac{1}{(x^2+1)(x^2-1)}\,dx$$

$$\frac{1}{(x^2+1)(x+1)(x-1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+1} + \frac{D}{x-1}$$

$$1 = (Ax+B)(x^2-1) + C(x^2+1)(x-1) + D(x^2+1)(x+1)$$

$x=1$: $1 = D \cdot 4$ $\qquad D = 1/4$

$x=-1$: $1 = C \cdot (-4)$ $\qquad C = -1/4$

$$1 = (Ax+B)(x^2-1) + \frac{1}{4}(x^2+1)\left(\underbrace{(x+1)-(x-1)}_{2}\right)$$

$A=0$: $1 = B(x^2-1) + \frac{1}{4}(x^2+1)\cdot 2$

$B = -1/2$

$$\frac{1}{(x^2+1)(x+1)(x-1)} = \frac{-1/2}{x^2+1} + \frac{-1/4}{x+1} + \frac{1/4}{x-1}$$

$$\int \frac{1}{x^4-1}\,dx = \frac{-1}{2}\arctan(x) - \frac{1}{4}\ln|x+1| + \frac{1}{4}\ln|x-1| + c$$

$$= \underline{\frac{-1}{2}\arctan x + \frac{1}{4}\ln\left|\frac{x-1}{x+1}\right| + c}$$

$$I_n = \int \frac{1}{(1+x^2)^n}\,dx \qquad n \geq 2$$

$$1 = (1+x^2) - x^2$$

$$I_n = \int \frac{(1+x^2) - x^2}{(1+x^2)^n}\,dx = \underbrace{\int \frac{1}{(1+x^2)^{n-1}}\,dx}_{I_{n-1}} - \int \underset{u}{x} \cdot \underbrace{\frac{x}{(1+x^2)^n}}_{v'}\,dx$$

$$\left(\frac{1}{U^{n-1}}\right)' = \left(U^{-(n-1)}\right)' = \frac{-(n-1)}{U^n}$$

$$u' = 1 \qquad v = \frac{-1/(2(n-1))}{(1+x^2)^{n-1}}$$

$$I_n = I_{n-1} - \left(\frac{x}{(1+x^2)^{n-1}} \cdot \frac{-1}{2(n-1)} - \int \frac{-1/(2(n-1))\,1}{(1+x^2)^{n-1}}\,dx\right)$$

$$= I_{n-1} + \frac{1}{2(n-1)} \cdot \frac{x}{(1+x^2)^{n-1}} - \frac{1}{2(n-1)} I_{n-1}$$

$$I_n = \frac{2n-3}{2n-2} I_{n-1} + \frac{1}{2(n-1)} \frac{x}{(1+x^2)^{n-1}}$$ Rekursiv formel.

$$I_2 = \frac{1}{2} I_1 + \frac{1}{2\cdot 1} \frac{x}{(1+x^2)} = \underline{\frac{1}{2} \arctan x + \frac{1}{2} \frac{x}{1+x^2}}$$

$$I_3 = \frac{6-3}{6-2} I_2 + \frac{1}{4} \frac{x}{(1+x^2)^2}$$

$$= \frac{3}{4}\left(\frac{1}{2} \arctan x + \frac{1}{2} \frac{x}{1+x^2}\right) + \frac{1}{4} \frac{x}{(1+x^2)^2}$$

$$\int \frac{1}{(1+x^2)^3} dx = \frac{3}{8} \arctan x + \frac{3}{8} \cdot \frac{x}{1+x^2} + \frac{1}{4} \frac{x}{(1+x^2)^2}$$

Integral på formen $\int \frac{x}{(1+x^2)^n} dx$, $n \geq 2$

løses ved : bruk av substitusjon

$1+x^2 = u$ $\qquad u' = 2x$

$$\int \frac{\frac{1}{2} du}{u^n} = \frac{-1}{2(n-1)} \frac{1}{u^{n-1}} + c$$

$$\int \frac{x}{(1+x^2)^n} dx = \frac{-1}{2(n-1)(1+x^2)^{n-1}} + c \text{ , } n \geq 2.$$