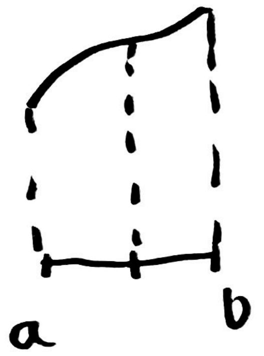
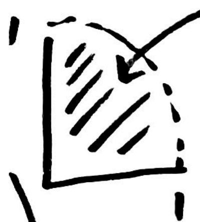
31.03 Simpson's metode 17.4

Deler opp intervallet i dobbeltintervaller



$$\int_a^b f(x)\,dx \sim S_2 = \frac{f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)}{6}(b-a)$$

vekting $\underline{\frac{1}{6}, \frac{4}{6}, \frac{1}{6}}$.



$$4\int_0^1 \sqrt{1-x^2}\,dx = \pi$$

$$S_2 = 4\left(\frac{1}{6}\left(1 + 4\cdot\sqrt{1-\left(\tfrac{1}{2}\right)^2} + 0\right)\cdot 1\right)$$

bredde

$$= \frac{4}{6}\left(1 + 4\sqrt{\tfrac{3}{4}}\right)$$

$$= \frac{2}{3}\left(1 + 2\sqrt{3}\right) \sim 2.97\ldots$$

$\uparrow$

bredde

$$= \frac{1}{3}\left(1 + \sqrt{3} + 4\left(\sqrt{\frac{15}{16}} + \sqrt{\frac{7}{16}}\right)\right)$$

$$= \frac{1}{3}\left(1 + \sqrt{3} + \sqrt{15} + \sqrt{7}\right) \sim \underline{3.083}$$

Implementerer Simpson's metode i Python.

Vekting $\frac{1}{6}(1\ 4\ 2\ 4\ 2\ 4 \cdots \cdots 2\ 4\ 1$

$f\,a$ $f(a+d)$ $f(a+2d)$ $f\,b$

n doble intervaller

$m = 2n$

$b - a$

$$S = \frac{1}{6}\Big[-f(a) + f(b) + 4\Big(\sum_{i=0}^{n-1} f(\ldots$$

$$+ 2 \sum^{n-1} f(a + 2id \ldots$$

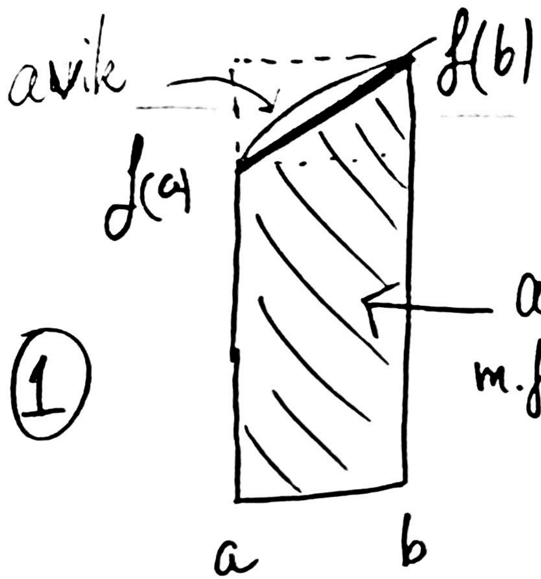
11.03
2022

Numerisk integrasjon

Fausk

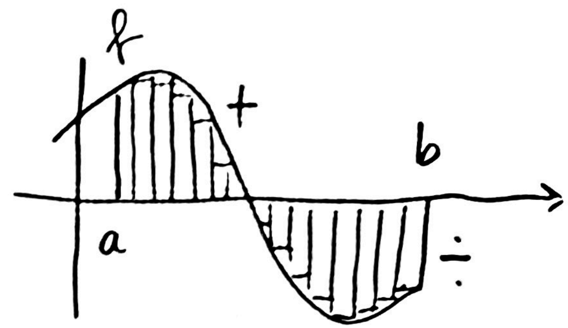
Trapesmetoden

1



areal $(b-a)\left(\frac{f(b)+f(a)}{2}\right)$
m.fortegn

Trapesmetoden gir eksakt verdi
når $f(x)$ er en lineær funksjon
$f(x) = ax + b$ (polynom av grad 1)

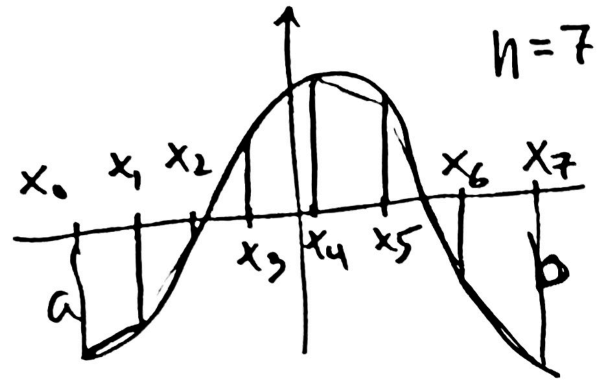


$$\int_a^b f\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{b-a}{n} \cdot f(x_i)$$

$$a + (i-1)\frac{b-a}{n} \le x_i \le a + i\left(\frac{b-a}{n}\right)$$

f kontinuerlig i $[a, b]$

2



$n=7$

$x_0 = a$

$x_1 = a+d$

$x_2 = a+2d$

$\vdots$

$\underline{x_i = a + i \cdot d}$

så

$x_n = a + n\frac{b-a}{n} = \underline{b}$

$d = \frac{b-a}{n}$ bredden til hvert av trapesene.

T_n tilnærming til $\int_a^b f(x)\,dx$ med n like intervaller.

$$T_n = d\left(\frac{f(x_0)+f(x_1)}{2} + \frac{f(x_1)+f(x_2)}{2} + \cdots + \frac{f(x_{n-1})+f(x_n)}{2}\right)$$

$$= d\left(\frac{f(x_0)}{2} + f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + \frac{f(x_n)}{2}\right)$$

Resultat $\left|\int_a^b f(x)\,dx - T_n\right| \leq \frac{M_2 (b-a)^3}{n^2} = M_2\left(\frac{b-a}{n}\right)^2(b-a)$

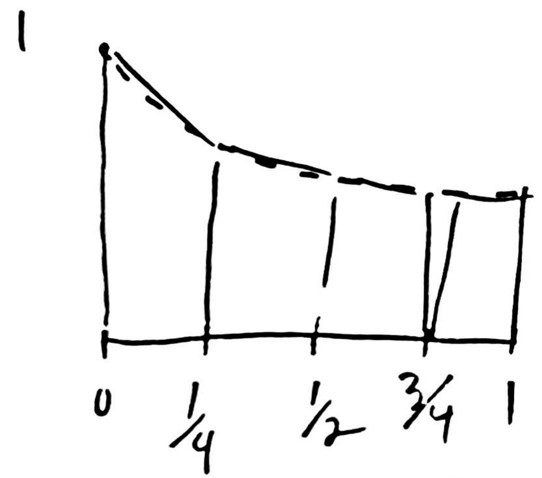
$M_2 = \max_{x\in[a,b]} |f''(x)|$

~~opg~~ Estimer $\int_0^1 \frac{1}{1+x^2}\,dx$ (eksakt verdi $\frac{\pi}{4}$)

eks.

med trapesmetoden og 4 intervaller.

(3)



$d = \frac{b-a}{4} = \frac{1}{4}$

$$T_4 = \frac{1}{4}\left(\frac{1}{2}f(0) + f(\tfrac{1}{4}) + f(\tfrac{1}{2}) + f(\tfrac{3}{4}) + \frac{1}{2}f(1)\right)$$

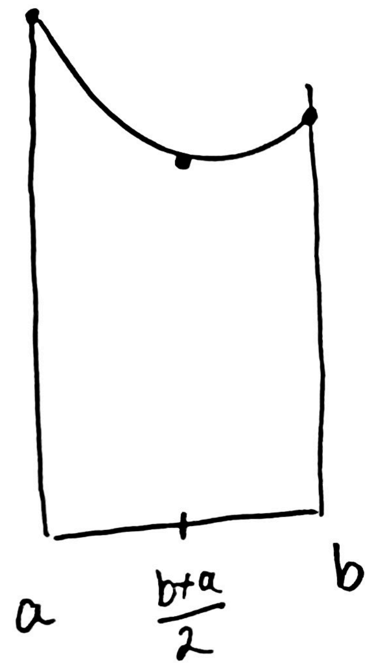
$$= \frac{1}{4}\left(\frac{1}{2}\cdot 1 + \frac{1}{1+(1/4)^2} + \frac{1}{1+(1/2)^2} + \frac{1}{1+(3/4)^2} + \frac{1}{4}\right)$$

$$= \frac{1}{4}\left(\frac{3}{4} + \frac{4^2}{4^2+1} + \frac{4^2}{4^2+4} + \frac{4^2}{4^2+9}\right)$$

$$\sim 0.78279\ldots$$ (tilnærming til en brøk)

④

Simpsons metode



Vekting 1 4 1

$$S_2 = \frac{f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)}{6} \cdot (b-a)$$

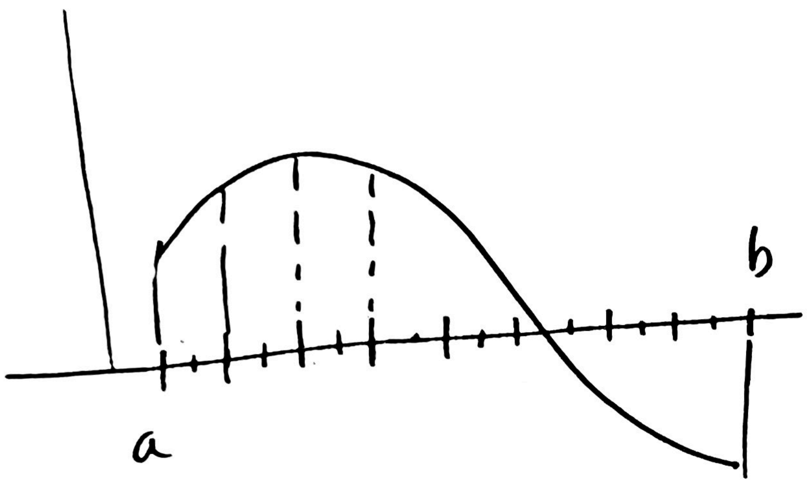
$$= \frac{f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)}{3} \cdot \left(\frac{b-a}{2}\right)$$

$$d = \frac{b-a}{2}$$

S_2 gir eksakt integral for 2.grads uttrykk.

Tilstrekkelig å sjekke for $f(x) = x^2$

5



m dobbelintervaller

$n = 2m$ intervaller

$$S_n = \frac{1}{3}\Big(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 4f(x_{2n-1}) + f(x_{2n}) \Big) \left(\frac{b-a}{n}\right)$$

141
 141
  141 ... etc

vekting 1 4 2 4 2 4 2 4 ... 4 1

Result: $$\left| \int_a^b f(x)\,dx - S_n \right| \leq \frac{M_4}{180} \frac{(b-a)^5}{n^4}$$

$$\int_a^b x^2\,dx = \frac{x^3}{3}\Big|_a^b = \underline{\frac{1}{3}(b^3-a^3)}$$

$$S_2 = \frac{b-a}{6}\Big(a^2 + \underbrace{4\cdot\Big(\frac{b+a}{2}\Big)^2}_{(b+a)^2} + b^2\Big) = \frac{b-a}{6}\big(a^2+a^2+2ab+b^2+b^2\big)$$

⑥

$$= \frac{b-a}{6}\,2\,(a^2+ab+b^2) = \frac{1}{3}(b-a)(a^2+ab+b^2)$$

$$= \frac{1}{3}\big(ba^2+ab^2+b^3-a^3-a^2b-ab^2\big)$$

$$= \frac{1}{3}(b^3-a^3)$$

så $\int_a^b x^2\,dx = S_2$

sjekk gjerne at $\int_a^b x^3\,dx = S_2$

Simpsons metode gir eksakt verdi for alle 3. grads polynomer!