27 mars 2023 17.5 Substitusjon (variabelskifte)

Kjerneregelen for derivasjon

$$\Big(F(\underbrace{u(x)}_{\text{kjerne}})\Big) = F'(u(x)) \cdot u'(x)$$

$$\frac{d}{dx} F(u(x)) = \frac{d}{du} F(u) \cdot \frac{du}{dx}$$

$$\frac{d}{dx}(1+x^3)^6 = 6(1+x^3)^5 \cdot (1+x^3)'$$

$$= 6(1+x^3)^5 \cdot 3x^2$$

$$= \underline{18x^2(1+x^3)^5}$$

Så $\int 18x^2(1+x^3)^5\,dx = \underline{(1+x^3)^6 + C}$

$$\int f(u)\,\underbrace{u'(x)\,dx}_{du} = \int f(u)\,du + C$$

$$\text{"}\,u'dx = \frac{du}{dx}\,dx = du\,\text{"}$$

$$\int \frac{x}{\sqrt{1-x^2}}\,dx$$

$$= \int x \cdot (1-x^2)^{-1/2}\,dx$$

$$= \int \left(\frac{-1}{2}u'\right) u^{-1/2}\,dx$$

$$= \frac{-1}{2}\int u^{-1/2}\,du$$

$$= \frac{-1}{2} \cdot \frac{u^{1/2}}{1/2} + C = \underline{-\sqrt{1-x^2} + C}$$

prøver

$$u = 1-x^2$$

$$u' = -2x$$

$$\frac{-1}{2}u' = x$$

oppg $\int \sin x \cdot \cos^2 x\,dx = \int \sin x\,(\cos x)^2\,dx$

prøver $u = \cos x$

$$u' = -\sin x$$

$$\int (-u') u^2 \, dx = -\int u^2 \, du$$
$$= -\frac{u^3}{3} + C$$
$$= -\frac{(\cos x)^3}{3} + C$$
$$= \underline{-\frac{1}{3}\cos^3(x) + C}$$

$$\int \frac{\sin x}{\cos x} dx = \int \tan x \, dx$$

$$\int \sin x \cdot (\cos x)^{-1} dx$$

$$\int -u' \cdot u^{-1} dx = -\int \frac{1}{u} du$$
$$= \underline{-\ln|\cos x| + C}$$

samme
substitusjon
som ovenfor

$$\int \sin^3 x \, dx$$

hint: Pytagoras

$$\sin^2 x + \cos^2 x = 1$$

$$\sin^2 x = 1 - \cos^2 x$$

$$\int \sin x \,(1 - \cos^2 x)\, dx$$

$$u = \cos x$$

$$u' = -\sin x$$

$$\int (-u')(1 - u^2)\, dx = -\int (1 - u^2)\, du$$

$$= -\left(u - \frac{u^3}{3}\right) + C$$

$$= \underline{-\left(\cos x - \frac{\cos^3 x}{3}\right) + C}$$

Hva er $\int \frac{1}{\sin x}\, dx$?

$$\int \sin^2 x \, dx = \int \sin x \frac{\sin x \, dx}{\pm\sqrt{1-\cos^2 x}}$$

$u = \cos x$

$u' = -\sin x$

$$= -\int \sqrt{1-u^2} \, du \quad \text{(fortegn ?)}$$

Bruker trig. identiteter: $\cos(2x) = \cos^2 x - \sin^2 x$

$$1 - \sin^2 x$$

$$= 1 - 2\sin^2 x$$

Så $\sin^2 x = \dfrac{1-\cos(2x)}{2}$

Derfor er $\int \sin^2 x \, dx = \int \frac{1}{2}(1-\cos(2x))\,dx$

$$= \frac{1}{2}\left(x - \int \cos(2x)\,dx\right) = \underline{\frac{1}{2}\left(x - \frac{1}{2}\sin(2x)\right) + C}$$

$u = \cos x \qquad 0 \leq x \leq \pi$

Så $\sin x \geq 0$ så $\sin x = \sqrt{1-\cos^2 x}$

$$-\int \sqrt{1-u^2}\, du = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C$$

$$= \underline{\frac{1}{2}\arccos(u) - \frac{1}{4}\cdot 2u\sqrt{1-u^2} + C}$$

Linear substitusjon

$u = ax + b \quad , \quad u' = a$

$$\int f(ax+b)\,dx = \int \frac{1}{a}\, \underbrace{a\, f(ax+b)}_{u' \cdot f(u)}\, dx$$

$$= \underline{\frac{1}{a}\int f(u)\, du}$$

$$\int (1+x^5)^2 \, dx$$

$1+x^5 = u$

$u' = 5x^4$

$x^5 = u-1$

$x^4 = (u-1)^{4/5}$

$$\int \underbrace{\frac{}{5x^4}}_{}^{u'} \cdot \frac{1}{5(u-1)^{4/5}} \, u^2 \, dx = \int \frac{u^2}{5(u-1)^{4/5}} \, du$$

vanskeligere!

$du = 5x^4 \, dx$

$\frac{1}{5x^4} du = dx$

vanlig feil:

$$\cancel{\frac{1}{5x^4} \int u^2 \, du}$$

skal være $\int \frac{1}{5x^4} u^2 \, du$

$$\int \frac{1}{(2x-3)}\,dx$$

$2x-3=u$

$u'=2$

$du=2dx$

$dx=\frac{1}{2}du$

$$= \frac{1}{2}\int \frac{1}{u}\,du$$

$$= \frac{1}{2}\ln|u| + C$$

$$= \underline{\frac{1}{2}\ln|2x-3| + C}$$

oppg

$$\int t\sqrt{t+3}\,dt$$

hint $u=t+3$

$u'=1$

$du=dt$

$t=u-3$

$$= \int (u-3)u^{1/2}\,du$$

$$= \int u^{3/2} - 3u^{1/2}\,du$$

$$= \frac{u^{5/2}}{5/2} - \frac{3u^{3/2}}{3/2} + C = \frac{2}{5}(t+3)^{5/2} - 2(t+3)^{3/2} + C$$

$$= \underline{\left(\frac{2}{5}\cdot(t+3)^2 - 2(t+3)\right)\sqrt{t+3} + C}$$

$$\int_a^b u'(x) f(u(x))\, dx = \int_{u(a)}^{u(b)} f(u)\, du$$

$$F'(u) = f(u)$$

$$\int u'(x) f(u(x))\, dx = F(u(x)) + C$$

Én antiderivert til $u' f(u)$.

$$\int_a^b u'(x) f(u(x))\, dx = F(u(x)) \Big|_a^b = F(u(b)) - F(u(a))$$

Dette er lik $\int_{u(a)}^{u(b)} f(u)\, du$

$$= F(u) \Big|_{u(a)}^{u(b)}.$$

Eks. $\int_0^1 (3x-1)^4\, dx$ La $u = 3x - 1$

$u' = 3$

$$\int (3x-1)^4 dx = \int \frac{1}{3} u^4 du$$

$$= \frac{1}{3} \frac{u^5}{5} + C = \underline{\frac{1}{15}(3x-1)^5 + C}$$

Sà $\int_0^1 (3x-1)^4 dx = \frac{1}{15}(3x-1)^5 \Big|_0^1$

I

$$= \frac{1}{15}\left(2^5 - (-1)^5\right) = \underline{\frac{33}{15}}$$

II $\int_0^1 (3x-1)^4 dx = \int_{u(0)}^{u(1)} \frac{1}{3} u^4 du = \frac{1}{3}\int_{-1}^{2} u^4 du$

$$= \frac{1}{3} \cdot \frac{1}{5} u^5 \Big|_{-1}^{2} = \underline{\frac{33}{15}}.$$

$$\int_0^{\pi} \sin x \, e^{\cos x} \, dx$$

$$= \int_0^{\pi} (-u') e^{u} \, dx$$

$$= -\int_1^{-1} e^{u} \, du$$

$$= \int_{-1}^{1} e^{u} \, du = e^{u} \Big|_{-1}^{1} = \underline{e - 1/e} \sim \underline{2.3504\ldots}$$

$u = \cos x$

$u' = -\sin x$

$\cos 0 = 1$

$\cos \pi = -1$

opg $$\int_0^1 \frac{e^{-x+1}}{2+e^{-x}} \, dx$$

$$= \int_0^1 \frac{(-e^{-x})(-e)}{2+e^{-x}} \, dx$$

$u = e^{-x}$

$u' = -e^{-x}$

$e^{-x+1} = e^{-x} \cdot e^{1}$

$= e \cdot e^{-x}$

$u(0) = e^{0} = 1$

$u(1) = e^{-1} = 1/e$

$$\int_{1}^{1/e} (-e)\frac{1}{2+u}\,du = e\int_{1/e}^{1} \frac{1}{2+u}\,du$$

$$= e\ln(2+u)\Big|_{1/e}^{1}$$

$$= e\left(\ln(3) - \ln(2+1/e)\right)$$

$$= \underline{e\ln\left(\frac{3}{2+1/e}\right)}.$$