13.03

Uendelige følger og rekker

$a_1, a_2, \ldots, a_n, \ldots$

$1, 2, 3, \ldots, n, \ldots$ divergerer

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots$ konvergere til 0

$1, -1, 1, -1, \ldots, (-1)^{n+1}, \ldots$ divergerer.

$\lim_{n \to \infty} a_n = a$ hvis a_n nermer seg a når $n \to \infty$

("$\epsilon - \delta$ formulering"
For alle $\epsilon > 0$ så finnes det et nat tall N slik at $|a_n - a| < \epsilon$ når $n \geq N$)

opg

* $a_n = \dfrac{n}{n+2}$ konverger følgen? Ja, den konverger til $\underline{1}$

$\dfrac{1}{3}, \dfrac{1}{2}, \dfrac{3}{5}, \dfrac{2}{3}, \ldots$

* $b_n = \dfrac{n}{\sqrt{n}+1}$ — " — ? Nei, den divergerer

$$\frac{n}{n+2} = \frac{n+2-2}{n+2} = 1 - \frac{2}{n+2} \to 1 \quad \text{når } n \to \infty$$

$$\frac{n}{\sqrt{n}+1} \underset{n \text{ stor}}{\sim} \frac{n}{\sqrt{n}} = \sqrt{n}$$

$n \geq 2$

$$\frac{n}{(\sqrt{n}+1)} \frac{(\sqrt{n}-1)}{(\sqrt{n}-1)} = \frac{n(\sqrt{n}-1)}{n-1}$$

$$= \left(\frac{n}{n-1}\right)(\sqrt{n}-1) = \left(1+\frac{1}{n-1}\right)(\sqrt{n}-1)$$

Uendelige rekker

$$\underbrace{a_1 + a_2 + \cdots + a_n}_{S_n \text{ delsum.}} + \cdots$$

Uendelig følge av delsummer $S_1, S_2, S_3, \ldots, S_n, \ldots$

Rekken konvergerer og har sum S hvis følgen av delsummer konvergerer til S.

$$\sum_{n=1}^{\infty} \underbrace{\frac{1}{n(n+1)}}_{\frac{1}{n} - \frac{1}{n+1}} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{30} + \cdots$$

m-te delsum $S_m = \sum_{n=1}^{m} \frac{1}{n(n+1)} = 1 - \frac{1}{m+1}$

Følgen av delsummer konverger til 1

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

Uendelig geometrisk rekke

$$\underbrace{1 + x + x^2 + x^3 + \cdots + x^n}_{S_n = \begin{cases} \frac{x^{n+1}-1}{x-1} & x \neq 1 \\ n+1 & x = 1 \end{cases}} + \cdots$$

$x \geq 1$ divergere

$|x| < 1$ konvergere til $\frac{1}{1-x}$

$|x| < 1$ da vil $x^n \to 0$ når $n \to \infty$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{(x-1)}(x^{n+1}-1) = \frac{-1}{x-1} = \underline{\frac{1}{1-x}}$$

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = \frac{1}{1-1/2} = \frac{1}{1/2} = 2.$$

$$1 + 0.99 + 0.99^2 + 0.99^3 + \cdots = \frac{1}{1-0.99} = \frac{1}{0.01} = \underline{100}$$

$$0.99^{10} \sim 0.904\ldots$$

$$0.99^{100} \sim 0.366\ldots$$

Hva er summen?

opg $1 + 0.01 + 0.0001 + 0.000001 + \cdots$

geometrisk rekke med kvotient $\frac{1}{100}$

konvergerer til $\frac{1}{1-1/100} = \frac{100}{100-1} = \underline{\frac{100}{99}}$

$$1.131313\ldots = 1.\underline{13}$$

$$= 1 + 13(0.01 + 0.0001 + \cdots)$$

$$= 1 + 13 \cdot 0.01\left(1 + \frac{1}{100} + \frac{1}{100^2} + \cdots\right) = 1 + 13 \cdot \frac{1}{100} \cdot \frac{100}{99}$$

$$= 1 + \frac{13}{99} = \frac{99+13}{99} = \underline{\frac{112}{99}}$$

Hvis $a_1 + a_2 + a_3 + \cdots$ konvergerer, da må $a_n \to 0$ når $n \to \infty$

$$a_n = S_n - S_{n-1}$$

Hvis $S_n \to S$ da vil

$$|a_n| = |S_n - S_{n-1}| = |S_n - S + S - S_{n-1}|$$
$$\leq \underbrace{|S_n - S|}_{\to 0} + \underbrace{|S_{n-1} - S|}_{\to 0}$$

derfor må $a_n \to 0$.

Selv om $a_n \to 0$ behøver ikke grensen konvergere.

$$S_n = \sum_{i=1}^{n} \frac{1}{\sqrt{i}} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}}$$

$a_n = \frac{1}{\sqrt{n}} \to 0$ når $n \to \infty$

$S_n \geq$ #ledd $\cdot$ (minste verdi til leddene)

$n \cdot \frac{1}{\sqrt{n}} = \sqrt{n}$

Derfor divergerer $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots$

Harmoniske rekken $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ divergerer

$\sum_{n=1}^{\infty} \frac{1}{n}$

($\sum_{n=1}^{\infty} \frac{1}{n^p}$ konvergerer for $p > 1$)

$$1 + \overbrace{\frac{1}{2} + \frac{1}{3}}^{\leq 1} + \overbrace{\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}}^{\leq 1} + \frac{1}{8} + \overbrace{\frac{1}{9} + \cdots + \frac{1}{2^n}}^{\leq 1}$$

$$1 \quad \frac{1}{2} \quad \underbrace{\frac{1}{3} + \frac{1}{4}}_{\geq 2 \cdot \frac{1}{4} = \frac{1}{2}} \quad \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{\geq 4 \cdot \frac{1}{8} = \frac{1}{2}}$$

$$\geq 1 + \underbrace{\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots \frac{1}{2}}_{n} = \underline{1 + \frac{n}{2}}$$

$$1 + \frac{n}{2} \leq \sum_{i=1}^{2^n} \frac{1}{i} \leq 1 + n - 1 = n \qquad n \geq 2.$$

$$6 \leq \sum_{i=1}^{1024} \frac{1}{i} \leq 10$$

$$11 \leq \sum_{i=1}^{2^{20}} \frac{1}{i} \leq 20$$

mer enn 1 million ledd.

13 mars. 23 Lån L

Nedbetales på A år. årlige avdrag B faste

årlig rente r $(5\% = 0.05)$

$A=1$ $B = L(1+r)$

L, $L(1+r) - B$, $(L(1+r)-B)(1+r) - B$, $L(1+r)^3 - B(1+r)^2 - B(1+r) - B$, $\dots$ $\underbrace{L(1+r)^A - \left(B(1+r)^{A-1} + \dots + B(1+r) + B\right)}_{0}$ lånet er nedbetalt.

$$L(1+r)^A = B\Big(\underbrace{1 + (1+r) + \dots + (1+r)^{A-1}}_{\frac{(1+r)^{A-1+1}-1}{(1+r)-1}}\Big) \qquad \frac{(1+r)^{A-1+1}-1}{(1+r)-1} = \frac{(1+r)^A - 1}{r}$$

$$\underline{B = L\frac{r(1+r)^A}{(1+r)^A - 1}}$$

Månedlige avdrag

$A \rightsquigarrow A \cdot 12$, $r \rightsquigarrow (1+r)^{1/12} - 1$

$$\underline{B = L\left((1+r)^{1/12} - 1\right)\frac{(1+r)^A}{(1+r)^A - 1}}$$

Oppg geometrisk rekke

$$6 + 2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \cdots$$

konvergens? sum?

Kvotienten er $\frac{2}{6} = \frac{1}{3}$.

$$6\left(1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \cdots\right)$$

$$= 6 \cdot \frac{1}{1 - 1/3} = 6 \cdot \frac{1}{2/3} = 6 \cdot \frac{3}{2} = \underline{9}$$

Oppg. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - + \cdots$

"alternerende rekke"

(Faktisk er summen lik $\ln 2 \sim 0.69$)

Konvergerer rekken? Ja legger samma ledd parvis

$$\frac{1}{2} + \frac{1}{12} + \frac{1}{30} + \cdots + \frac{1}{2n(2n-1)} + \cdots = \sum_{n=1}^{\infty} \frac{1}{2n(2n-1)} \leq \sum_{1}^{\infty} \frac{1}{n(n+1)} = 1$$