27.02.23

oblig 5. Typiske feil: Rot når man skal finne alle løsningen til en Δ-likning i en oppgitt mengde.

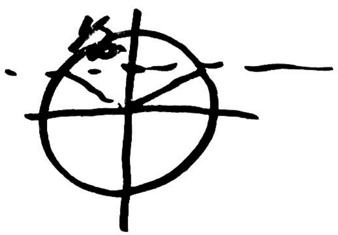
Eksempel

$$\sin(2x+1) = \tfrac{1}{2} \qquad x \in \langle 0, 5 \rangle$$

Finn alle løsningene eksakt.

1. $\sin(v) = \tfrac{1}{2}$
2. $2x+1 = v$

1.



$$v_1 = \frac{\pi}{6} + 2\pi \cdot n$$

$$v_2 = \frac{5\pi}{6} + 2\pi \cdot n$$

2. $2x+1 = \frac{\pi}{6} + 2\pi \cdot n$ så $x_1 = \frac{\pi}{12} - \frac{1}{2} + \pi \cdot n$

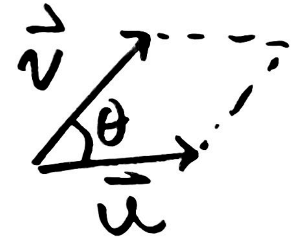
$= \frac{5\pi}{6} + 2\pi \cdot n$ $\qquad x_2 = \frac{5\pi}{12} - \frac{1}{2} + \pi \cdot n$

$\left(\frac{\pi}{12} \approx 0.26\right)$ Løsningene er: $x = \frac{\pi}{12} - \frac{1}{2} + \pi = \underline{\frac{13\pi}{12} - \frac{1}{2}}$,

$$\left(x_2 \approx \underbrace{1.3 - \tfrac{1}{2}}_{0.8} + \pi \cdot n\right)$$

$$\underline{x = \frac{5\pi}{12} - \frac{1}{2}} \quad \text{og} \quad \underline{\frac{17\pi}{12} - \frac{1}{2}}$$

14.5 **Vektorprodukt (i rommet)**



$\vec{u} \times \vec{v}$ en vektor

$|\vec{u} \times \vec{v}| = |\vec{u}| \cdot |\vec{v}| \cdot \sin(\theta)$

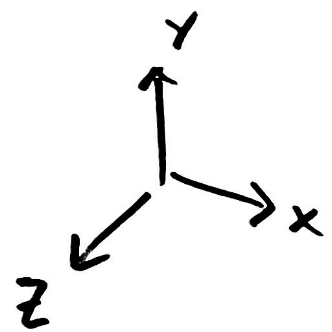
(arealet til parallellogrammet utspent av $\vec{u}$ og $\vec{v}$)

$\vec{u} \times \vec{v}$ ortogonal til $\vec{u}$ og $\vec{v}$.
planet utspent av $\vec{u}$ og $\vec{v}$

$\vec{u}, \vec{v}, \vec{u} \times \vec{v}$ høyrehandsystem.

$\vec{u} \times \vec{v}$ er lineær i både $\vec{u}$ og $\vec{v}$.
(litt arbeid å syne)

$$\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}.$$



på koordinatform.

$$[x_1, y_1, z_1] \times [x_2, y_2, z_2] = \begin{vmatrix} \vec{e_1} & \vec{e_2} & \vec{e_3} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

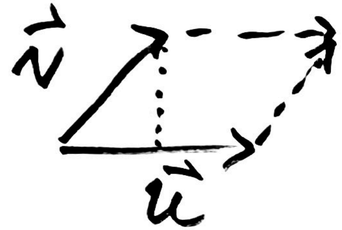
$$= \left[\begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}, - \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix}, \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} \right].$$

$$[-1, 2, -1] \times [3, 4, -7]$$

$$= \begin{vmatrix} \vec{e}_1, & \vec{e}_2 & \vec{e}_3 \\ -1 & 2 & -1 \\ 3 & 4 & -7 \end{vmatrix} = [2(-7) - 4(-1), -((-1)(-7) - (-1)3), -1(4) - 2 \cdot 3] = [-10, -10, -10] = \underline{-10\,[1, 1, 1]}$$

Finn arealet til parallellogrammet utspent av

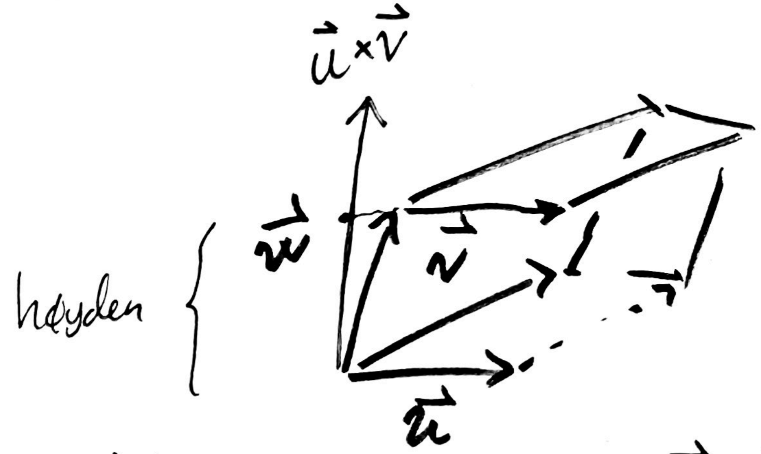
$\underbrace{[-1, 2, -1]}_{\vec{u}}$ og $\underbrace{[3, 4, -7]}_{\vec{v}}$



$$A = |-10\,[1,1,1]| = 10\,|[1,1,1]|$$

$$= 10\sqrt{1^2 + 1^2 + 1^2} = \underline{10\sqrt{3}}$$

14.6 Parallellepipeder



$$\vec{w}\cdot(\vec{u}\times\vec{v}) = \pm\text{høyde}\cdot\underbrace{|\vec{u}\times\vec{v}|}_{\text{areal til grunnflaten.}}$$

$$= \pm \text{Volum til parallellepipedet.}$$

$$\vec{w}\cdot\begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ & \vec{u} & \\ & \vec{v} & \end{vmatrix} = \begin{vmatrix} \vec{w} \\ \vec{u} \\ \vec{v} \end{vmatrix}$$ determinanten til $\vec{w}$, $\vec{u}$ og $\vec{v}$.

Finn volumet til parallellepipedet utspent av

$$\underset{\vec{u}}{[-1,2,-1]}, \quad \underset{\vec{v}}{[3,4,-7]} \quad \text{og} \quad \underset{\vec{w}}{[2,3,-1]}$$

$$V = \text{abs}(\vec{w} \cdot (\vec{u} \times \vec{v})) = \text{abs}((\vec{u} \times \vec{v}) \cdot \vec{w})$$

$$= \text{abs}(-10[1,1,1] \cdot [2,3,-1])$$

$$= \text{abs}(-10(1 \cdot 2 + 1 \cdot 3 + 1 \cdot (-1))) = \text{abs}(-10 \cdot 4)$$

$$= \underline{\underline{40}}$$

—

Finn volumet til parallellepipedet utspent av

$$\vec{u} = [0, 1, 2]$$
$$\vec{v} = [3, 4, 1]$$
$$\vec{w} = [5, 4, 3].$$

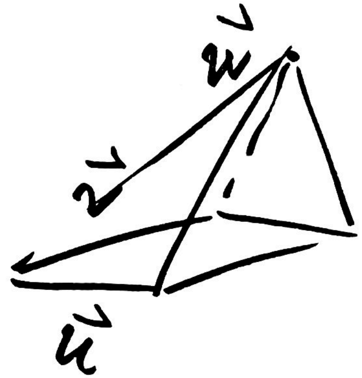
$$V = \text{abs} \begin{vmatrix} \vec{u} \\ \vec{v} \\ \vec{w} \end{vmatrix}$$

$$V = \text{abs} \begin{vmatrix} 0 & 1 & 2 \\ 3 & 4 & 1 \\ 5 & 4 & 3 \end{vmatrix} = \text{abs}\left(0 - 1 \begin{vmatrix} 3 & 1 \\ 5 & 3 \end{vmatrix} + 2 \begin{vmatrix} 3 & 4 \\ 5 & 4 \end{vmatrix}\right)$$

$$= \text{abs}\left(-(9-5) + 2(4(3-5))\right) = \text{abs}(-4 + 2(-8))$$

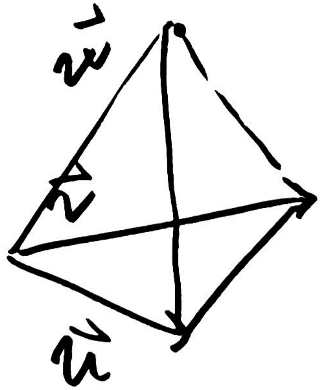
$$= \text{abs}(-20) = \underline{20}$$

Pyramide



$$V = \frac{1}{3} \text{abs} \begin{vmatrix} \vec{u} \\ \vec{v} \\ \vec{w} \end{vmatrix}$$

Tetraeder



$$V = \frac{1}{6} \text{abs} \begin{vmatrix} \vec{u} \\ \vec{v} \\ \vec{w} \end{vmatrix}$$

Determinanter : $\begin{vmatrix} \vec{u} \\ \vec{v} \\ \vec{w} \end{vmatrix}$ skalar.

to like

antisymmetrisk i vektorene $\left(\Rightarrow \begin{vmatrix} \vec{u} \\ \vec{u} \\ \vec{v} \end{vmatrix} = 0 \right)$

lineær i hver av vektorene.

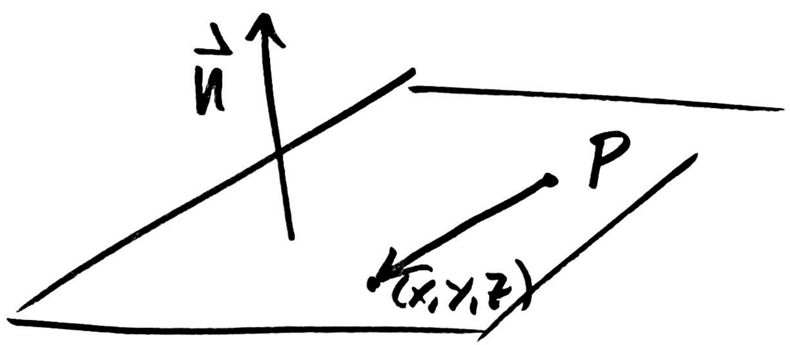
$$\begin{vmatrix} 9 & 27 & 81 \\ 0 & 4 & 8 \\ -7 & -7 & -7 \end{vmatrix} = - \begin{vmatrix} 4[0,1,2] \\ 9[1,3,9] \\ -7[1\ 1\ 1] \end{vmatrix}$$

$$= -4 \cdot 9 \cdot (-7) \begin{vmatrix} 0 & 1 & 2 \\ 1 & 3 & 9 \\ 1 & 1 & 1 \end{vmatrix} = 4 \cdot 7 \cdot 9 \begin{vmatrix} 0 \quad 1 \quad 2 \\ [1,1,1] + [0,2,8] \\ [1,1,1] \end{vmatrix}$$

$$= 4 \cdot 7 \cdot 9 \begin{vmatrix} 0 & 1 & 2 \\ 0 & 2 & 8 \\ 1 & 1 & 1 \end{vmatrix} = 4 \cdot 7 \cdot 9 \left(0 - 1 \begin{vmatrix} 0 & 8 \\ 1 & 1 \end{vmatrix} + 2 \begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} \right)$$

$$= 4 \cdot 7 \cdot 9 (8 - 4) = \underline{16 \cdot 7 \cdot 9}$$

14.7 Plan i rommet



(x, y, z) i planet $\Leftrightarrow$

$\overrightarrow{P(x, y, z)}$ ortogonal til $\vec{n}$

$\vec{n} \cdot \overrightarrow{P(x,y,z)} = 0$

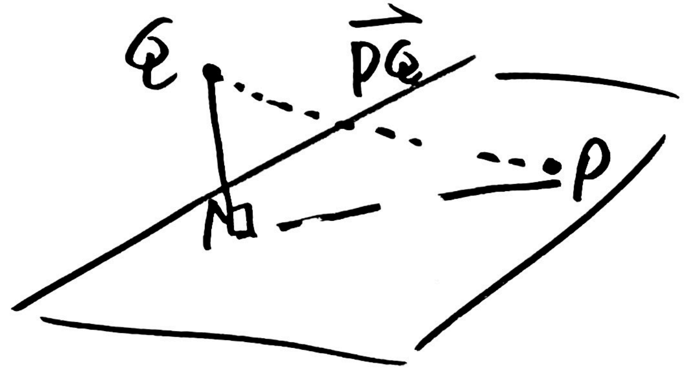
$\vec{n}\,(\underbrace{\overrightarrow{O(x,y,z)}}_{[x,y,z]} - \overrightarrow{OP}) = 0 \quad \Leftrightarrow \quad \vec{n} \cdot [x, y, z] = \vec{n} \cdot \overrightarrow{OP}$

$[a, b, c] = \vec{n}$

$d = \vec{n} \cdot \overrightarrow{OP}$.

$$\boxed{ax + by + cz = d}$$

Løsningene er planet.



Avstand mellom plan og punkt.
(Kortest mulig avstand)

Komponenten til $\vec{PQ}$ langs $\vec{n}$ er korteste vektor fra planet til Q.

Så avstanden fra Q til planet er

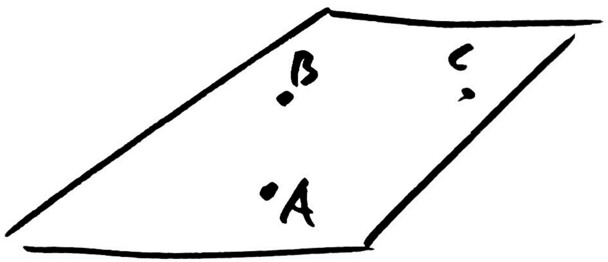
$$\left| \frac{\vec{PQ} \cdot \vec{n}}{|\vec{n}|^2} \vec{n} \right| = \frac{|\vec{PQ} \cdot \vec{n}|}{|\vec{n}|}.$$

Finn avstanden fra planet $2x - y + 3z = 7$ til $Q(1,1,1)$

$$\vec{PQ} \cdot \vec{n} = (\vec{OQ} - \vec{OP}) \cdot \vec{n} = [1,1,1] \cdot \vec{n} - \overbrace{\vec{OP} \cdot \vec{n}}^{7}$$

$$= 4 - 7 = -3.$$

Avstanden er $\dfrac{|-3|}{|[2,-1,3]|} = \dfrac{3}{\sqrt{4+1+9}} = \underline{\dfrac{3}{\sqrt{14}}}$



Tre punkt i rommet som ikke ligger på en linje utspenner et plan.

$A(1,2,-5)$ $\quad B(-3,4,2)$ $\quad C(0,3,1)$

$\vec{u} = \vec{CA}$ og $\vec{v} = \vec{CB}$ utspenner planet

$\vec{u} \times \vec{v}$ står normalt på planet

$\vec{u} = \vec{OA} - \vec{OC} = [1,2,-5] - [0,3,1] = [1,-1\ -6]$

$\vec{v} = \vec{OB} - \vec{OC} = [-3,4,2] - [0,3,1] = [-3,1\ 1]$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 1 & -1 & -6 \\ -3 & 1 & 1 \end{vmatrix} = [5, 17, -2]$$

$= \vec{n}$

Planet:

$$\vec{n} \cdot [x, y, z] = \vec{n} \cdot \underbrace{[0, 3, 1]}_{\overrightarrow{OC}}$$

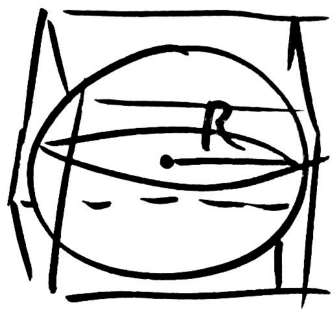
$$= [5, 17, -2] \cdot [0, 3, 1]$$

$$= 3 \cdot 17 + (-2 \cdot 1) = \underline{49}$$

$$\underline{5x + 17y - 2z = 49}$$

oblig#5 opg 13

ØVING



$$\frac{\text{Volum kule}}{\text{Volum kube}} = \frac{\pi}{6}.$$

Kule m. radius R

$$V_{kule} = \frac{4\pi}{3} R^3.$$

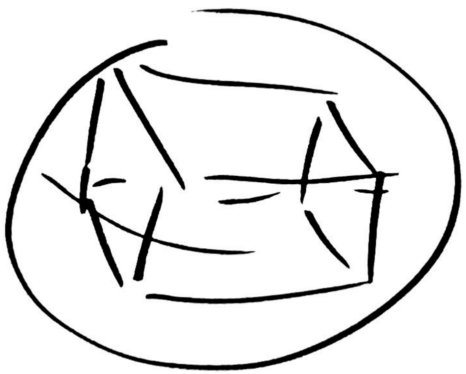
Minste kube som inneholder kulen har sider av lengde 2R.

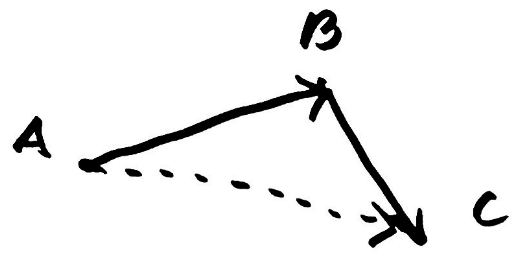
$$V_{kube} = (2R)^3 = 8R^3$$

$$\frac{V_{kule}}{V_{kube}} = \frac{4\pi R^3/3}{8 \cdot R^3} = \frac{4\pi}{8 \cdot 3} = \frac{\pi}{6}$$

$$\sim 0.5235\ldots$$

oppgaven



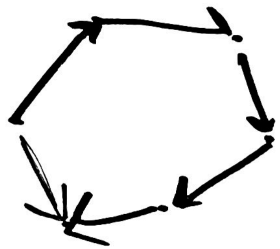


$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

$$5\overrightarrow{AB} - 5\overrightarrow{CB} = 5(\overrightarrow{AB} \underbrace{- \overrightarrow{CB}}_{\overrightarrow{BC}})$$

$$= 5(\overrightarrow{AB} + \overrightarrow{BC})$$

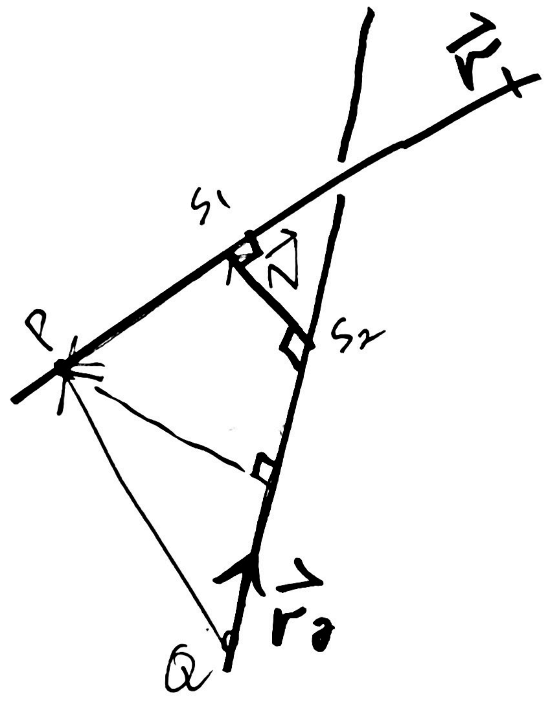
$$= \underline{5\overrightarrow{AC}}$$



$$5\overrightarrow{AB} - 3\overrightarrow{CB} + 2\overrightarrow{BA} = 3(\overrightarrow{AB} \underbrace{- \overrightarrow{CB}}_{\overrightarrow{BC}}) + 2\overrightarrow{AB} + 2\underbrace{\overrightarrow{BA}}_{-\overrightarrow{AB}}$$

$$= 3\overrightarrow{AC} + 2\underbrace{\overrightarrow{AA}}_{\vec{0}} = \underline{3\overrightarrow{AC}}$$

12.



$\vec{r}_1 \times \vec{r}_2$ ortogonal til både $\vec{r}_1$ og $\vec{r}_2$ så korteste linjestykke mellom de to linjene er parallelt til $\vec{r}_1 \times \vec{r}_2$.

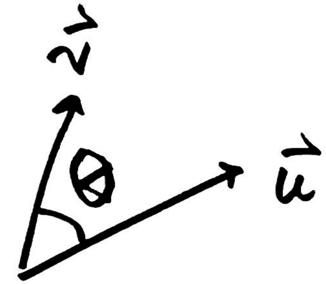
$\vec{v}$ er komponenten til $\vec{QP}$ langs $\vec{r}_1 \times \vec{r}_2$.

$$\vec{QP} = \vec{OP} - \vec{OQ}$$

$$= \underbrace{\vec{QS_2}}_{\text{parallell til } r_2} + \underbrace{\vec{S_2S_1}}_{\vec{v}} + \underbrace{\vec{S_1P}}_{\text{parallell til } \vec{r}_1}$$

3

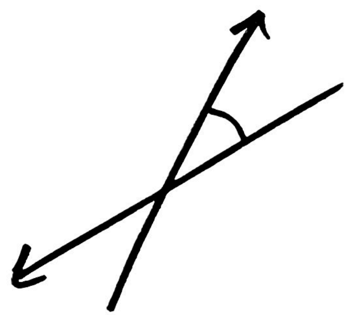
$\vec{u} = [1, 1, 2]$ $\quad \vec{v} = [3, -2, -1]$



$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta$$

$$3 + (-2) + (-1)2 = -1$$

$$-1 = \underbrace{\sqrt{1+1+4}}_{|\vec{u}|}\ \underbrace{\sqrt{9+4+1}}_{|\vec{v}|}$$



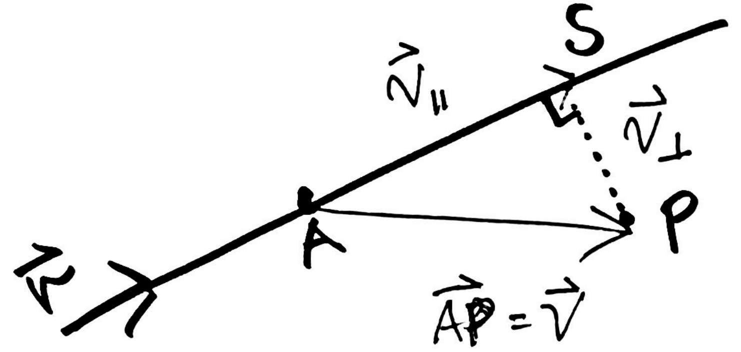
$$\cos\theta = \frac{-1}{\sqrt{6}\sqrt{14}}$$

$$\theta = \arccos\left(\frac{-1}{\sqrt{84}}\right) \sim 96.26°$$

Vinkelen mellom linjene parallelle til $\vec{u}$ og $\vec{v}$

er $180° - 96.26° = \underline{83.74°}$

6



$$\vec{v}_{\parallel} = \frac{\vec{v} \cdot \vec{r}}{|\vec{r}|} \cdot \frac{\vec{r}}{|\vec{r}|}$$

$$= \frac{\vec{v} \cdot \vec{r}}{|\vec{r}|^2} \, \vec{r} \, .$$

$$\vec{v}_{\parallel} + \vec{v}_{\perp} = \vec{v}$$

$$\vec{v}_{\perp} = \vec{v} - \vec{v}_{\parallel}$$

$$\left(\begin{aligned} \vec{OP} - \vec{OS} &= \vec{SP} = \vec{v}_{\perp} \\ \vec{OS} &= \vec{OP} - \vec{v}_{\perp} \end{aligned} \right)$$

(Korteste) avstand mellom P og linjen er

$$|\vec{v}_{\perp}|$$

Linje $\begin{cases} A(1,-1,2) \\ \vec{r} = [1,2,3] \end{cases}$

punkt $P(6,5,4)$

$$\vec{v} = \vec{AP} = \vec{OP} - \vec{OA} = [6,5,4] - [1,-1,2] = [5,6,2]$$

$$\vec{v} \cdot \vec{r} = [5,6,2] \cdot [1,2,3] = 5 + 12 + 6 = 23$$

$$\vec{v}_{\parallel} = \frac{\vec{v} \cdot \vec{r}}{|\vec{r}|^2} \vec{r} = \frac{23}{14} [1,2,3] . \quad \left(|\vec{r}|^2 = \sqrt{1+2^2+3^2} = \sqrt{14} \right)$$

$$\vec{v}_{\perp} = \vec{AP} - \vec{v}_{\parallel} = \vec{v} - \vec{v}_{\parallel} = [5,6,2] - \frac{23}{14} [1,2,3]$$

$$\vec{v}_{\perp} = \left[\frac{47}{14}, \frac{38}{14}, \frac{-41}{14} \right]$$

Korteste avstand $|\vec{v}_{\perp}| = \frac{1}{14} \left| [47, 38, -41] \right| = \frac{1}{14} \sqrt{5334}$

$\sim 5.2167\ldots$

Punktet S på linjen nermest P er: $\vec{OS} = \vec{OA} + \vec{v}_{\parallel} \ldots$

Alternativ:

Parametrisera linjen

$$[x,y,z] = \vec{OA} + t\vec{r}$$
$$= [1,-1,2] + t[1,2,3].$$

Punktet S på linjen er nemest linjen når

$$\vec{SP} \perp \vec{r} \iff \vec{SP}\cdot\vec{r} = 0$$
$$(\vec{OP} - \vec{OS})\cdot\vec{r} = 0$$

$(\vec{r} = [1,2,3])$

$$\vec{OS}\cdot\vec{r} = \vec{OP}\cdot\vec{r}.$$
$$([1,-1,2] + t[1,2,3])\cdot\vec{r} = [6,5,4]\cdot\vec{r}$$
$$(1+2(-1)+3\cdot 2) + t|\vec{r}|^2 = 6\cdot 1 + 5\cdot 2 + 4\cdot 3$$
$$t = \frac{1}{|\vec{r}|^2}(28-5) = \frac{23}{14}$$
$$\vec{OS} = \vec{OA} + t\vec{r} = [1,-1,2] + \frac{23}{14}[1,2,3] = \frac{1}{14}[37,32,97]$$

$S(37/14, 32/14, 97/14)$