PICARD GROUPS OF DERIVED CATEGORIES

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ABSTRACT. We investigate the group $\mathrm{Pic}(\mathcal{D}_{\mathcal{M}})$ of isomorphism classes of invertible objects in the derived category of $\mathcal{O}$-modules for a commutative unital ringed Grothendieck topos $(\mathcal{E}, \mathcal{O})$ with enough points. When the ring $\mathcal{O}_p$ has connected prime ideal spectrum for all points p of $\mathcal{E}$ we show that $\mathrm{Pic}(\mathcal{D}_{\mathcal{M}})$ is naturally isomorphic to the Cartesian product of the Picard group of $\mathcal{O}$-modules and the additive group of continuous functions from the space of isomorphism classes of points of $\mathcal{E}$ to the integers $\mathbb{Z}$. Also, for a commutative unital ring R, the group $\mathrm{Pic}(\mathcal{D}_R)$ is isomorphic to the Cartesian product of $\mathrm{Pic}(R)$ and the additive group of continuous functions from $\mathrm{spec}\, R$ to the integers $\mathbb{Z}$.

1. Introduction

There has recently been much interest in Picard groups of monoidal categories [2, 6, 7, 8, 9, 11, 12, 14]. In particular, the Picard group of the derived category for a commutative unital ring has been calculated in some cases [5, 12, 14]. The goal of this paper is to calculate the Picard group of the unbounded derived category for any commutative unital ring.

More generally, we calculate the Picard group of the derived category for any commutative unital ringed Grothendieck topos $(\mathcal{E}, \mathcal{O})$ with enough points [4, 6.4.1, 11.1.1]. We assume that there is a small site $\mathcal{C}$ such that $\mathcal{E}$ is equivalent to the category of sheaves of sets on the site $\mathcal{C}$, and we let $\mathrm{pt}(\mathcal{E})$ denote the set of isomorphism classes of points of $\mathcal{E}$. We let $\mathcal{M}$ denote the category of left $\mathcal{O}$-modules in $\mathcal{E}$. The derived category $\mathcal{D}_{\mathcal{M}}$ is obtained from the category of cochain complexes of left $\mathcal{O}$-modules by formally inverting cochain maps that induce an isomorphism on cohomology.

The derived category $\mathcal{D}_{\mathcal{M}}$ is a symmetric tensor category. The derived tensor product $M^\bullet \otimes^{\mathrm{L}}_{\mathcal{O}} N^\bullet$ in $\mathcal{D}_{\mathcal{M}}$ is obtained as the ordinary

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tensor product of complexes if $M^\bullet$ or $N^\bullet$ is replaced by a flat resolution [13]. In section 2 we recall the definition of flat complexes, and construct flat resolutions for all complexes of $\mathcal{O}$-modules.

Let $(\mathcal{T}, \otimes, I)$ be a symmetric tensor category. An object X in $\mathcal{T}$ is defined to be invertible if there exists an object Y such that $X \otimes Y$ is isomorphic to the unit object I. The Picard group $\mathrm{Pic}(\mathcal{T})$ of $\mathcal{T}$ is defined to be the group of isomorphism classes of invertible objects in $\mathcal{T}$ (under the assumption that it is a set). The multiplication in $\mathrm{Pic}(\mathcal{T})$ is the tensor product, and the class of the unit object is the unit element.

Let R be a commutative unital ring, and let $\mathcal{D}_R$ denote the derived category of complexes of left R-modules. For a topological space T let $C(T)$ denote the additive group of continuous functions from T to the integers $\mathbb{Z}$. In section 3 we prove that there is a natural split short exact sequence

$$0 \to \mathrm{Pic}(R) \to \mathrm{Pic}(\mathcal{D}_R) \xrightarrow{\Psi} C(\mathrm{spec}\, R) \to 0.$$

Let $M^\bullet$ be an invertible complex of R-modules. The function $\Psi(M^\bullet)$ sends a prime ideal $\mathfrak{p}$ to the unique integer n such that $\mathrm{H}^n(M^\bullet)_\mathfrak{p} \neq 0$.

In section 4 we prove that, if the ring $\mathcal{O}_p$ has a connected prime ideal spectrum for all points p of $\mathcal{E}$, then there is a natural split short exact sequence

$$0 \to \mathrm{Pic}(\mathcal{M}) \to \mathrm{Pic}(\mathcal{D}_\mathcal{M}) \xrightarrow{\Psi} C(\mathrm{pt}(\mathcal{E})) \to 0.$$

In general, let $C(\mathrm{spec}\, \mathcal{O})$ be the sheaf associated to the presheaf which maps an object X to $C(\mathrm{spec}\, \mathcal{O}(X))$. We prove that there is a natural split short exact sequence

$$0 \to \mathrm{Pic}(\mathcal{M}) \to \mathrm{Pic}(\mathcal{D}_\mathcal{M}) \xrightarrow{\tilde{\Psi}} \Gamma(C(\mathrm{spec}\, \mathcal{O})) \to 0$$

where Γ is the global sections functor.

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2. The derived tensor product and flat resolutions

We recall the following definition from [13, 5.1].

Definition 2.1. A complex $F^\bullet$ is flat if for all acyclic complexes $A^\bullet$ the complex $F^\bullet \otimes_\mathcal{O} A^\bullet$ is acyclic. A flat resolution of a complex $M^\bullet$ is a flat complex $F^\bullet$ and a map $F^\bullet \to M^\bullet$ inducing an isomorphism on cohomology.

Note that flat complexes are sometimes called K-flat or q-flat complexes to avoid confusion with complexes of flat objects.

For an object X in the site $\mathcal{C}$ let X also denote the sheaf of sets in $\mathcal{E}$ represented by the object X. Let $j_X : \mathcal{E}|X \to \mathcal{E}$ be the localization map [4, 5.2]. The $\mathcal{O}$-module $j_{X!}j_X^*\mathcal{O}$ is called the free $\mathcal{O}$-module generated by X, and we denote it by $\mathcal{O}_X$ [4, 11.3.1].

For any point p of $\mathcal{E}$ there is a canonical isomorphism of stalks

$$(\mathcal{O}_X)_p \cong \oplus_{X_p}\mathcal{O}_p$$

where the sum is over the set X_p [4, 11.3.5]. Hence $(\mathcal{O}_X)_p$ is a flat $\mathcal{O}_p$-module for all points p of $\mathcal{E}$. Since $\mathcal{E}$ has enough points it follows that $\mathcal{O}_X$ is a flat $\mathcal{O}$-module.

There is a natural isomorphism [4, 11.3.3]

$$\hom_{\mathcal{M}}(\mathcal{O}_X, M) \cong \hom_{\mathcal{E}}(X, M) = M(X).$$

For an element $m \in M(X)$ let m also denote the unique morphism $\mathcal{O}_X \to M$ in $\mathcal{M}$. Let $\mathrm{ad} : 1 \to j_X^* j_{X!}$ be the unit map of the $(j_{X!}, j_X^*)$-adjunction. The map

$$j_X^* m \circ \mathrm{ad}(j_X^*\mathcal{O}) : j_X^*\mathcal{O} \to j_X^*\mathcal{O}_X \to j_X^* M$$

evaluated at 1_X sends the unit element in $j_X^*\mathcal{O}(1_X) = \mathcal{O}(X)$ to the element m in $M(X) = j_X^* M(1_X)$.

Since the site $\mathcal{C}$ is small we can construct the following epic map of $\mathcal{O}$-modules

$$F(M) = \oplus_{X\in\mathcal{C}} \oplus_{m\in M(X)\setminus\{0\}} \mathcal{O}_X \to M.$$

When $M = 0$ we set $F(M) = 0$. Since $\mathcal{E}$ has enough points this gives a canonical resolution of any $\mathcal{O}$-module M by flat $\mathcal{O}$-modules

$$\cdots \xrightarrow{d^{-3}} F(\ker(d^{-1})) \xrightarrow{d^{-2}} F(\ker(d^0)) \xrightarrow{d^{-1}} F(M) \xrightarrow{d^0} M \to 0.$$

Let us denote this resolution $\mathcal{F}^\bullet(M)$. For any map $g : M \to N$ there is a canonical map $F(g) : F(M) \to F(N)$ such that $d^0 \circ F(g) = g \circ d^0$. This gives a canonical map $\mathcal{F}^\bullet(g) : \mathcal{F}^\bullet(M) \to \mathcal{F}^\bullet(N)$ of the resolutions. Hence any bounded above cochain complex of $\mathcal{O}$-modules

$$\cdots \to M^{n-2} \to M^{n-1} \to M^n \to 0$$

has a resolution which is (the totalization of)

$$\cdots \to \mathcal{F}^\bullet(M^{n-2}) \to \mathcal{F}^\bullet(M^{n-1}) \to \mathcal{F}^\bullet(M^n) \to 0.$$

We call this resolution the standard resolution. For each point p of $\mathcal{E}$ the complex $\mathcal{F}^\bullet(M)_p$ is a complex of free $\mathcal{O}_p$-modules. Hence the standard resolution localized at a point p is a bounded above complex of free $\mathcal{O}_p$-modules, which implies that it is a flat complex of $\mathcal{O}_p$-modules [13, 3.2, 5.8]. Since $\mathcal{E}$ has enough points it follows that the standard resolution is a flat complex of $\mathcal{O}$-modules. Hence all bounded above cochain complexes have flat resolutions.

For an $\mathcal{O}$-module M and an integer n let $M[n]$ denote the complex which is M in degree n and trivial in all other degrees. In particular the map $M \mapsto M[0]$ defines a monomorphism $\mathrm{Pic}(\mathcal{M}) \to \mathrm{Pic}(\mathcal{D}_{\mathcal{M}})$. Let $\tau_{\leq n}M^\bullet$ denote the sub-complex of $M^\bullet$ which is 0 in degrees above n, the kernel of $d : M^n \to M^{n+1}$ in degree n, and agrees with $M^\bullet$ in degrees below n. The map $\tau_{\leq n}M^\bullet \to M^\bullet$ induces an isomorphism on cohomology groups in degrees less than or equal to n. The following lemma is from [13, 3.3].

Lemma 2.2. *Let $M^\bullet$ be a cochain complex of $\mathcal{O}$-modules. Fix an integer k. There exists a sequence*

$$F_k^\bullet \to F_{k+1}^\bullet \to F_{k+2}^\bullet \to \cdots$$

of flat complexes of $\mathcal{O}$-modules mapping into $M^\bullet$ such that $F_n^q = 0$ for all $q > n$ and $\mu_n : F_n^\bullet \to \tau_{\leq n}M^\bullet$ induces an isomorphism on cohomology for each $n \geq k$. Let $F^\bullet = colim(F_n^\bullet)$. The map $F^\bullet \to M^\bullet$ is a flat resolution of $M^\bullet$.

Proof. For the given k, let $F_k^\bullet \to \tau_{\leq k}M^\bullet$ be the standard resolution. We now construct $F_n^\bullet$ inductively. Assume that $\mu_{n-1} : F_{n-1}^\bullet \to \tau_{\leq n-1}M^\bullet$ is given. Let μ'_{n-1} denote the composite of μ_{n-1} with the inclusion $\tau_{\leq n-1}M^\bullet \to \tau_{\leq n}M^\bullet$. Construct the following diagram:

$$\begin{array}{ccccccccc}
F_{n-1}^\bullet & \xrightarrow{\mu'_{n-1}} & \tau_{\leq n}M^\bullet & \longrightarrow & C_{\mu'_{n-1}} & \longrightarrow & F_{n-1}^\bullet[-1] & \xrightarrow{\mu'_{n-1}[-1]} & \tau_{\leq n}M^\bullet[-1] \\
 & & & & \simeq\uparrow & & \Vert & & \uparrow \\
 & & & & Q^\bullet & \xrightarrow{g} & F_{n-1}^\bullet[-1] & \longrightarrow & C_g\,.
\end{array}$$

The cone $C_{\mu'_{n-1}}$ is a complex which is 0 in degrees above n, and we let $Q^\bullet \to C_{\mu'_{n-1}}$ be the standard resolution. In particular Q^i is 0 for $i > n$.

Define $F_n^\bullet$ to be $C_g[1]$ and let $\mu_n : C_g[1] \to \tau_{\leq n}M^\bullet$ be defined by the diagram above. The map μ_n induces an isomorphism on cohomology. By construction $F_n^i = 0$ for $i > n$, and since both $Q^\bullet$ and $F_{n-1}^\bullet$ are flat complexes of $\mathcal{O}$-modules $F_n^\bullet$ is a flat complex of $\mathcal{O}$-modules. Note that μ'_{n-1} is the composite map

$$F_{n-1}^\bullet \to F_n^\bullet \xrightarrow{\mu_n} \tau_{\leq n}M^\bullet.$$

Since cohomology commutes with filtered colimits $F^\bullet = \mathrm{colim}(F_n^\bullet) \to M^\bullet$ is a cohomology isomorphism, and since each $F_n^\bullet$ is a flat complex $F^\bullet$ is also a flat complex. □

Remark 2.3. A ring R (resp. an R-module) is a ringed space (resp. module over the ringed space) when the site is the category with only one object and one morphism.

3. The Picard group of the derived category of R-modules

Let R be a commutative unital ring.

Lemma 3.1. *Let $M^\bullet$ be an invertible object in $\mathcal{D}_R$. Then there exists an integer m such that $H^q(M^\bullet) = 0$ for $q > m$ and $H^m(M^\bullet) \neq 0$ is a finitely generated R-module.*

Proof. Replace $M^\bullet$ by a flat resolution $F^\bullet$ as in Lemma 2.2. $R[0]$ is the unit object in $\mathcal{D}_R$. Let $u : R[0] \cong F^\bullet \otimes_R G^\bullet$ be an isomorphism in $\mathcal{D}_R$ where $G^\bullet$ is a flat complex. There is an isomorphism $v : G^\bullet \otimes_R F^\bullet \cong R[0]$ such that

$$F^\bullet \xrightarrow{u\otimes 1} F^\bullet \otimes_R G^\bullet \otimes_R F^\bullet \xrightarrow{1\otimes v} F^\bullet$$

is the identity map in $\mathcal{D}_R$.

By choosing a cycle representing the image of 1 under the isomorphism $R \cong \mathrm{H}^0(F^\bullet \otimes_R G^\bullet)$ we get a map of cochain complexes

$$\eta : R[0] \to F^\bullet \otimes_R G^\bullet$$

inducing the isomorphism on cohomology. Since $F^\bullet = \mathrm{colim}(F_n^\bullet)$ the map η factors through $F_n^\bullet \otimes_R G^\bullet$ for some integer n. By tensoring the commutative triangle

$$\begin{array}{ccc} & \nearrow & F_n^\bullet \otimes_R G^\bullet \\ & & \downarrow \\ R[0] & \xrightarrow{\simeq} & F^\bullet \otimes_R G^\bullet \end{array}$$

with $F^\bullet$ from the right and using the isomorphism $G^\bullet \otimes_R F^\bullet \cong R[0]$ in $\mathcal{D}_R$ we see that $F^\bullet$ is a retract of $F_n^\bullet$ in $\mathcal{D}_R$. Since $F_n^q = 0$ for $q > n$ and $M^\bullet$ has nontrivial cohomology we conclude that there is an integer m such that $\mathrm{H}^q(M^\bullet) = 0$ for $q > m$ and $\mathrm{H}^m(M^\bullet) \neq 0$. In particular $\mu_m : F_m^\bullet \to M^\bullet$ is a cohomology isomorphism.

Next we show that $F_m^\bullet$ is a retract in $\mathcal{D}_R$ of a bounded complex $B^\bullet$ which is a finitely generated R-module in each degree and satisfies $B^q = 0$ for $q > m$. As above there is a map of cochain complexes

$$\eta : R[0] \to F_m^\bullet \otimes_R G^\bullet$$

inducing an isomorphism on cohomology. Let $\eta(1) = \Sigma_{i,j\in\mathbb{Z}} f_{i,j} \otimes_R g_{i,j}$ where $f_{i,j} \in F_m^i$ and $g_{i,j} \in G^{-i}$ for all i and j, and almost all the $f_{i,j}$ are zero. Let $B^\bullet$ be the subcomplex of $F_m^\bullet$ generated by all the $f_{i,j}$. Since almost all the $f_{i,j}$ are zero, $B^\bullet$ is a bounded subcomplex of $F_m^\bullet$, so $B^q = 0$ for $q > m$, and B^n is a finitely generated R-module for each integer n.

The map η factors through $B^\bullet \otimes_R G^\bullet$. Tensoring the factorization with $F_m^\bullet$ from the right gives the commutative diagram

$$\begin{array}{ccccc} & & B^\bullet \otimes_R G^\bullet \otimes_R F_m^\bullet & \xrightarrow{\simeq} & B^\bullet \\ & \nearrow & \downarrow & & \downarrow \\ F_m^\bullet & \xrightarrow{\simeq} & F_m^\bullet \otimes_R G^\bullet \otimes_R F_m^\bullet & \xrightarrow{\simeq} & F_m^\bullet . \end{array}$$

Since both $F_m^\bullet$ and $G^\bullet$ are flat all the tensor products are derived tensor products. We get that $F_m^\bullet$, hence $M^\bullet$, is a retract of $B^\bullet$ in $\mathcal{D}_R$. In particular $\mathrm{H}^m(M^\bullet)$ is a retract of $\mathrm{H}^m(B^\bullet) = B^m/d(B^{m-1})$. Hence $\mathrm{H}^m(M^\bullet)$ is a finitely generated R-module. □

It is easy to see that if the cohomology of $M^\bullet$ is concentrated in one degree, say n, then $M^\bullet$ is isomorphic to $\mathrm{H}^n(M^\bullet)[n]$ in $\mathcal{D}_R$. The proof of the next result follows [5, V.3.3].

Proposition 3.2. *If R is a local ring then up to isomorphism the invertible objects in $\mathcal{D}_R$ are precisely the $R[n]$ for any integer n.*

Proof. The complex $R[n]$ is invertible for any n since $R[n] \otimes_R^{\mathrm{L}} R[-n] \cong R[0]$.

Let $M^\bullet \otimes_R^{\mathrm{L}} N^\bullet \cong R[0]$ in $\mathcal{D}_R$. Since the complexes $M^\bullet$ and $N^\bullet$ are bounded from above by Lemma 3.1, there is a convergent Künneth spectral sequence

$$E_2^{p,q} = \bigoplus_{i+j=q} \mathrm{Tor}_R^p(\mathrm{H}^i(M^\bullet), \mathrm{H}^j(N^\bullet)) \Rightarrow \mathrm{H}^{p+q}(M^\bullet \otimes_R^{\mathrm{L}} N^\bullet)$$

where the grading is so that Tor_R^p is zero for $p > 0$. Since $M^\bullet \otimes_R^L N^\bullet \cong R[0]$ the cohomology group $\mathrm{H}^r(M^\bullet \otimes_R^{\mathrm{L}} N^\bullet)$ is isomorphic to R when $r = 0$ and is zero when $r \neq 0$. If $\mathrm{H}^i(M^\bullet) = 0$ for $i > m$ and $\mathrm{H}^j(N^\bullet) = 0$ for $j > n$, the spectral sequence gives us that the Künneth map

$$\mathrm{H}^m(M^\bullet) \otimes_R \mathrm{H}^n(N^\bullet) \to \mathrm{H}^{m+n}(M^\bullet \otimes_R^{\mathrm{L}} N^\bullet)$$

is an isomorphism. If, further, $\mathrm{H}^m(M^\bullet) \neq 0$ and $\mathrm{H}^n(N^\bullet) \neq 0$, then by Lemma 3.1 both modules are finitely generated. One proves by induction on the number of generators and by Nakayama's lemma that $\mathrm{H}^m(M^\bullet) \otimes_R \mathrm{H}^n(N^\bullet)$ is also nonzero, hence is isomorphic to R. The ring R is local so up to isomorphism R is the only invertible R-module, hence $\mathrm{H}^m(M^\bullet) \cong R$ and $\mathrm{H}^n(N^\bullet) \cong R$. We show that $\mathrm{H}^i(M^\bullet) = 0$ whenever $i < m$. Assume by induction on $l \geq 1$ that $\mathrm{H}^i(M^\bullet) = 0$ for $m - l < i < m$ and that $\mathrm{H}^i(N^\bullet) = 0$ for $n - l < i < n$. We then have that

$$E_2^{0,-l} = \mathrm{H}^{m-l}(M^\bullet) \oplus \mathrm{H}^{n-l}(N^\bullet)$$

survives to the E_∞ term, hence $\mathrm{H}^{m-l}(M^\bullet)$ and $\mathrm{H}^{n-l}(N^\bullet)$ must both be zero.

Since the cohomology of $M^\bullet$ is concentrated in degree m where it is isomorphic to R, we get that $M^\bullet \cong R[m]$ in $\mathcal{D}_R$. □

There is a canonical Künneth homomorphism of degree 0 [1, IV.6.1]

$$\alpha : \mathrm{H}^*(F^\bullet) \otimes \mathrm{H}^*(G^\bullet) \to \mathrm{H}^*(F^\bullet \otimes G^\bullet).$$

The Künneth homomorphism is preserved by exact functors which commute up to isomorphism with the tensor product.

Lemma 3.3. *Let R be a commutative unital ring. If $M^\bullet$ is an invertible object in $\mathcal{D}_R$ then*

$$\oplus_{n\in\mathbb{Z}} H^n(M^\bullet)$$

is an invertible R-module.

Proof. Let $M^\bullet \otimes^{\mathrm{L}}_R N^\bullet \cong R[0]$ in $\mathcal{D}_R$. Localizing the Künneth homomorphism

$$\alpha : \mathrm{H}^*(M^\bullet) \otimes_R \mathrm{H}^*(N^\bullet) \to \mathrm{H}^*(M^\bullet \otimes^{\mathrm{L}}_R N^\bullet)$$

at a prime ideal $\mathfrak{p}$ in R gives us the Künneth homomorphism

$$\alpha_\mathfrak{p} : \mathrm{H}^*(M_\mathfrak{p}^\bullet) \otimes_{R_\mathfrak{p}} \mathrm{H}^*(N_\mathfrak{p}^\bullet) \to \mathrm{H}^*(M_\mathfrak{p}^\bullet \otimes^{\mathrm{L}}_{R_\mathfrak{p}} N_\mathfrak{p}^\bullet).$$

By Proposition 3.2 the Künneth homomorphism $\alpha_\mathfrak{p}$ is an isomorphism for all prime ideals $\mathfrak{p}$ in R, hence α is an isomorphism. This means that

$$\oplus_{i\in\mathbb{Z}} (\mathrm{H}^i(M^\bullet) \otimes_R \mathrm{H}^{-i}(N^\bullet)) \cong R$$

and

$$\oplus_{i+j=q} (\mathrm{H}^i(M^\bullet) \otimes_R \mathrm{H}^j(N^\bullet)) = 0$$

for all integers $q \neq 0$. Hence we get that

$$(\oplus_{i\in\mathbb{Z}} \mathrm{H}^i(M^\bullet)) \otimes_R (\oplus_{j\in\mathbb{Z}} \mathrm{H}^j(N^\bullet)) \cong R$$

and $\oplus_{i\in\mathbb{Z}} \mathrm{H}^i(M^\bullet)$ is an invertible R-module. □

Lemma 3.3 implies that for each prime ideal $\mathfrak{p}$ in R, there is an integer $\Psi(M^\bullet)(\mathfrak{p}) = n$ such that $\mathrm{H}^n(M^\bullet)_\mathfrak{p} \cong R_\mathfrak{p}$ and $\mathrm{H}^q(M^\bullet)_\mathfrak{p} = 0$ for all $q \neq n$. For a topological space T let $C(T)$ denote the additive group of continuous functions from T to the integers $\mathbb{Z}$ with the discrete topology.

We recall a lemma about idempotents.

Lemma 3.4. *There is a bijection between the open closed subsets of $\operatorname{spec} R$ and the idempotents in R. Let e_U denote the idempotent corresponding to the open closed set U. The bijection has the following properties.*

(1) *The subset* $\mathrm{spec}\, e_U R \subset \mathrm{spec}\, R$ *is* U. *The open closed set* U *is* $\{\mathfrak{p} \in \mathrm{spec}\, R \mid e_U R_{\mathfrak{p}} = R_{\mathfrak{p}}\}$. *If* $\mathfrak{q} \notin U$ *then* $e_U R_{\mathfrak{q}} = 0$.
(2) *The correspondence gives a natural isomorphism between the Boolean algebra of idempotents in* R *and the Boolean algebra of open closed subsets in* $\mathrm{spec}\, R$.

The second statement means that for any open closed sets U_1 and U_2, $e_{U_1 \cap U_2} = e_{U_1} e_{U_2}$, $e_{U_1 \cup U_2 - U_1 \cap U_2} = e_{U_1} + e_{U_2} - 2e_{U_1 \cap U_2}$, $e_X = 1$ and $e_\emptyset = 0$, and if $f : R \to S$ is a map of rings and U is an open closed set in $\mathrm{spec}\, R$ then $f(e_U^R) = e^S_{\mathrm{spec}(f)^{-1}(U)}$ in S.

Theorem 3.5. *Let* R *be a commutative unital ring. There is a natural split short exact sequence*

$$0 \to Pic(R) \to Pic(\mathcal{D}_R) \xrightarrow{\Psi} C(spec\, R) \to 0.$$

Proof. From [10, 4.10] the set of prime ideals such that $\mathrm{H}^n(M^\bullet)_{\mathfrak{p}} \cong R_{\mathfrak{p}}$ is an open closed set in $\mathrm{spec}\, R$. Hence $\mathfrak{p} \mapsto \Psi(M^\bullet)(\mathfrak{p})$ defines a map $\Psi : \mathrm{Pic}(\mathcal{D}_R) \to C(\mathrm{spec}\, R)$. It is clear from the proof of Proposition 3.2 that Ψ is a homomorphism. If $\Psi(M^\bullet) \equiv 0$ then $M^\bullet \cong \mathrm{H}^0(M^\bullet)[0]$ in $\mathcal{D}_R$, and since $\mathrm{H}^0(M^\bullet)$ is an invertible R-module $M^\bullet$ is in the image of $\mathrm{Pic}(R) \to \mathrm{Pic}(\mathcal{D}_R)$.

We construct a splitting of Ψ. The spectrum of R is a compact space so the image of any continuous function $f : \mathrm{spec}\, R \to \mathbb{Z}$ consists of a finite set of integers, say $n_1, \ldots, n_m$. The disjoint subsets $U_i = f^{-1}(n_i)$ of $\mathrm{spec}\, R$ are both open and closed, hence correspond to an orthogonal basis of idempotents $e_{U_1}, \ldots, e_{U_m}$ in R. Define the invertible complex $\Phi(f)$ to be $\oplus_{i=1}^m e_{U_i} R[n_i]$. By Lemma 3.4 the composite $\Psi \circ \Phi(f)$ is equal to f.

We now check that Φ is a homomorphism. Note that if $f \equiv 0$ then $\Phi(f) \cong R[0]$. For two finite open closed partitions $\{U_i\}$ and $\{V_i\}$ of $\mathrm{spec}\, R$ we have that

$$\oplus_{i=1}^N e_{U_i} R[n_i] \otimes_R \oplus_{j=1}^M e_{V_j} R[m_j] \cong \oplus_{i,j=1}^{N,M} e_{U_i \cap V_j} R[n_i + m_j].$$

Hence Φ is a homomorphism. It is easy to see that the split short exact sequence is natural. □

4. The Picard group of the derived category of $\mathcal{O}$-modules

Recall that $(\mathcal{E}, \mathcal{O})$ is a ringed topos with enough points, and that $\mathcal{M}$ is the category of left $\mathcal{O}$-modules.

Proposition 4.1. *Let $F^\bullet$ be an invertible object in the derived category of left $\mathcal{O}$-modules. Then*

$$\oplus_{n\in\mathbb{Z}} H^n(F^\bullet)$$

is an invertible $\mathcal{O}$-module.

Proof. Let $F^\bullet \otimes^{\mathrm{L}}_{\mathcal{O}} G^\bullet \cong \mathcal{O}$ in $\mathcal{D}_{\mathcal{M}}$. Assume that $F^\bullet$ is a flat complex of $\mathcal{O}$-modules as constructed in Lemma 2.2. The localization of $F^\bullet$ at a point p of $\mathcal{E}$ is then a flat complex of $\mathcal{O}_p$-modules. Taking the stalk at a point p of the Künneth homomorphism α gives

$$\alpha_p : \mathrm{H}^*(F^\bullet_p) \otimes_{\mathcal{O}_p} \mathrm{H}^*(G^\bullet_p) \to \mathrm{H}^*(F^\bullet_p \otimes^{\mathrm{L}}_{\mathcal{O}_p} G^\bullet_p)$$

in the category of $\mathcal{O}_p$-modules. By the proof of Lemma 3.3 α_p is an isomorphism for all points p. Since $\mathcal{E}$ has enough points the Künneth homomorphism α is an isomorphism. Hence we get as in the proof of Lemma 3.3 that

$$(\oplus_{i\in\mathbb{Z}} \mathrm{H}^i(F^\bullet)) \otimes_{\mathcal{O}} (\oplus_{j\in\mathbb{Z}} \mathrm{H}^j(G^\bullet)) \cong \mathcal{O}$$

and $\oplus_{i\in\mathbb{Z}} \mathrm{H}^i(F^\bullet)$ is an invertible $\mathcal{O}$-module. □

Theorem 4.2. *Let $(\mathcal{E}, \mathcal{O})$ be a commutative unital ringed Grothendieck topos with enough points such that for all points p of $\mathcal{E}$ the ring $\mathcal{O}_p$ has a connected prime ideal spectrum. Then there is a natural split short exact sequence*

$$0 \to Pic(\mathcal{M}) \to Pic(\mathcal{D}_{\mathcal{M}}) \xrightarrow{\Psi} C(pt(\mathcal{E})) \to 0.$$

Proof. Let $F^\bullet$ be an invertible complex in $\mathcal{D}_{\mathcal{M}}$. By Proposition 4.1 there exists an $\mathcal{O}$-module G such that

$$\oplus_{i\in\mathbb{Z}} \mathrm{H}^i(F^\bullet) \otimes_{\mathcal{O}} G \cong \mathcal{O}.$$

Let $A^i = \mathrm{H}^i(F^\bullet) \otimes_{\mathcal{O}} G$. Then $\oplus_{i\in\mathbb{Z}} A^i \cong \mathcal{O}$. Localized at a point p of $\mathcal{E}$ this gives $\oplus_{i\in\mathbb{Z}} A^i_p \cong \mathcal{O}_p$. From our assumptions on $\mathcal{O}_p$ there is an integer n_p such that $A^{n_p}_p \cong \mathcal{O}_p$ and $A^i_p = 0$ for $i \neq n_p$. Define $\Psi(F^\bullet)(p)$ to be n_p. Clearly Ψ is a natural homomorphism.

If $\Psi(F^\bullet) \equiv 0$, then $\mathrm{H}^0(F^\bullet) \otimes_{\mathcal{O}} G \cong \mathcal{O}$ and $\mathrm{H}^i(F^\bullet) = 0$ for all $i \neq 0$ so $F^\bullet \cong \mathrm{H}^0(F^\bullet)[0]$ in $\mathcal{D}_{\mathcal{M}}$. Hence the kernel of Ψ is the image of $\mathrm{Pic}(\mathcal{M}) \to \mathrm{Pic}(\mathcal{D}_{\mathcal{M}})$.

It remains to prove that Ψ takes values in $C(\mathrm{pt}(\mathcal{E}))$ and is split. We need to prove that if $A \oplus B \cong \mathcal{O}$ then $\{p \in \mathrm{pt}(\mathcal{E}) \mid A_p \cong \mathcal{O}_p\}$ is an open closed set in $\mathrm{pt}(\mathcal{E})$. Denote the terminal object in $\mathcal{E}$ by $\bullet$. The sheaf of sets $\bullet$ associates to every object in the site $\mathcal{C}$ the one-point set.

Let $1 : \bullet \to \mathcal{O}$ be the unit and $0 : \bullet \to \mathcal{O}$ the zero element. We can compose these two elements with the projection from $\mathcal{O} \cong A \oplus B$ to A.

There is an equalizer

$$S \longrightarrow \bullet \underset{0}{\overset{1}{\rightrightarrows}} A$$

in $\mathcal{E}$, and S is a subobject of $\bullet$. Points preserve limits so we get for each point p of $\mathcal{E}$ an equalizer

$$S_p \longrightarrow \bullet \underset{0}{\overset{1}{\rightrightarrows}} A_p$$

in the category of sets. Since $S_p \neq \emptyset$ if and only if $1 = 0$ in A_p it follows that

$$\{p \in \mathrm{pt}(\mathcal{E}) \,|\, A_p = 0\} = \{p \in \mathrm{pt}(\mathcal{E}) \,|\, S_p = \bullet\}$$

which by definition is an open set in $\mathrm{pt}(\mathcal{E})$ [4, 7.8]. The same argument applied to B shows that $\{p \in \mathrm{pt}(\mathcal{E}) \,|\, B_p = 0\}$ is also open in $\mathrm{pt}(\mathcal{E})$. Since $A_p \oplus B_p \cong \mathcal{O}_p$ and $\mathcal{O}_p$ has connected prime ideal spectrum exactly one of A_p and B_p is zero, so the two sets $\{p \in \mathrm{pt}(\mathcal{E}) \,|\, A_p = 0\}$ and $\{p \in \mathrm{pt}(\mathcal{E}) \,|\, B_p = 0\}$ are complements of each other in $\mathrm{pt}(\mathcal{E})$. Hence $\{p \in \mathrm{pt}(\mathcal{E}) \,|\, A_p = 0\}$ is an open closed set in $\mathrm{pt}(\mathcal{E})$.

We now construct a splitting of Ψ. Let f be a continuous function $\mathrm{pt}(\mathcal{E}) \to \mathbb{Z}$. For each integer n let S_n be the subobject of $\bullet$ corresponding to the open closed set $f^{-1}(n)$. For a subobject S in $\bullet$, let $\mathcal{O}_S$ be defined by $\mathcal{O}_S(X) = \mathcal{O}(X)$ if $S(X) = \bullet$ and $\mathcal{O}_S(X) = 0$ if $S(X) = \emptyset$ for $X \in \mathcal{C}$. By considering the zero and the unit maps $0, 1 : S_i \to \mathcal{O}_{S_i}$, using the evident isomorphism $\oplus_{i \in \mathbb{Z}} \mathcal{O}_{S_i} \to \mathcal{O}$ and our assumption on $\mathcal{O}_p$ it is easy to see that $(\mathcal{O}_{S_i})_p \cong \mathcal{O}_p$ if and only if $(S_i)_p = \bullet$. Define the complex $\Phi(f)$ to be $\mathcal{O}_{S_i}$ in degree i and to have trivial differentials. Since $\mathcal{O}_S \otimes_{\mathcal{O}} \mathcal{O}_T = \mathcal{O}_{S \cap T}$ for two subobjects S and T of $\bullet$, it follows that Φ is a homomorphism. In particular

$$\Phi(f) \otimes_{\mathcal{O}} \Phi(-f) \cong \Phi(0) \cong \mathcal{O}[0].$$

So Φ takes values in $\mathrm{Pic}(\mathcal{D}_{\mathcal{M}})$. Since $(\mathcal{O}_{S_i})_p \cong \mathcal{O}_p$ if and only if $(S_i)_p = \bullet$, the composite $\Psi \circ \Phi$ is the identity on $C(\mathrm{pt}(\mathcal{E}))$. It is easy to see that all three maps in the split short exact sequence are natural with respect to maps of ringed topoi. □

As a special case we get the following result.

Corollary 4.3. *Let $(X, \mathcal{O})$ be a locally ringed space with an action by a discrete group G [3]. There is a natural split short exact sequence*

$$0 \to Pic_G(X) \to Pic(\mathcal{D}_{sh_G(X)}) \xrightarrow{\Psi} C^G(X) \to 0.$$

Here $\mathrm{sh}_G(X)$ denotes the category of left G-$\mathcal{O}$-modules, $\mathrm{Pic}_G(X)$ denotes the group of isomorphism classes of invertible G-$\mathcal{O}$-modules, and $C^G(X)$ denotes the G-fixed subgroup of $C(X)$ [4, 8.4.1].

We now generalize the theorem to ringed topoi where the $\mathcal{O}_p$ are not necessarily connected. Define a sheaf of abelian groups $C(\operatorname{spec}\mathcal{O})$ by sheafifying the presheaf which maps an object $X \in \mathcal{C}$ to $C(\operatorname{spec}\mathcal{O}(X))$. Let $\Gamma(F) = \hom_{\mathcal{E}}(\bullet, F)$ denote the global sections functor.

Proposition 4.4. *There is a natural split short exact sequence*

$$0 \to Pic(\mathcal{M}) \to Pic(\mathcal{D}_{\mathcal{M}}) \xrightarrow{\tilde{\Psi}} \Gamma(C(spec\,\mathcal{O})) \to 0.$$

Proof. Let $F^\bullet$ be an invertible complex in $\mathcal{D}_{\mathcal{M}}$. From Proposition 4.1 $\oplus_{i\in\mathbb{Z}} \mathrm{H}^i(F^\bullet)$ is an invertible $\mathcal{O}$-module. Let G be an $\mathcal{O}$-module such that $\oplus_{i\in\mathbb{Z}} \mathrm{H}^i(F^\bullet) \otimes_{\mathcal{O}} G \cong \mathcal{O}$. The direct sum (resp. tensor product) in $\mathcal{M}$ is the sheafification of the presheaf direct sum (resp. tensor product). Since $\mathcal{O}$ is unital there is a covering $\{V_\gamma\}$ such that

$$\oplus_{i\in\mathbb{Z}} \mathrm{H}^i(F^\bullet)(V_\gamma) \otimes_{\mathcal{O}(V_\gamma)} G(V_\gamma) = (\oplus_{i\in\mathbb{Z}} \mathrm{H}^i(F^\bullet) \otimes_{\mathcal{O}} G)(V_\gamma) \cong \mathcal{O}(V_\gamma)$$

for each γ. Define $\psi_\gamma : \operatorname{spec}\mathcal{O}(V_\gamma) \to \mathbb{Z}$ by letting $\psi_\gamma(\mathfrak{p})$ be the unique integer i such that $\mathrm{H}^i(F^\bullet)(V_\gamma)_{\mathfrak{p}} \neq 0$. The maps $\{\psi_\gamma\}$ are compatible so they define an element $\tilde{\Psi}(F^\bullet)$ in $\Gamma(C(\operatorname{spec}\mathcal{O}))$. It is easy to see that the map $\tilde{\Psi}$ is a homomorphism.

We now construct a splitting Φ of $\tilde{\Psi}$. Given $f \in \Gamma(C(\operatorname{spec}\mathcal{O}))$, there exists a covering $\{V_\gamma\}$ and maps $f_\gamma : \operatorname{spec}\mathcal{O}(V_\gamma) \to \mathbb{Z}$ such that $\{f_\gamma\} = f$ in $\Gamma(C(\operatorname{spec}\mathcal{O}))$. By Lemma 3.4 there are unique idempotents $e^n_\gamma \in \mathcal{O}(V_\gamma)$ such that for each e^n_γ the subspace $\operatorname{spec}(e^n_\gamma\mathcal{O}(V_\gamma)) \subset \operatorname{spec}\mathcal{O}(V_\gamma)$ is $f^{-1}_\gamma(n)$. For a given n let $e^n = \{e^n_\gamma\}$. Then e^n is an idempotent element in $\Gamma(\mathcal{O})$, $e^n e^m = 0$ for $n \neq m$, and $\oplus_{i\in\mathbb{Z}} e^n\mathcal{O} \to \mathcal{O}$ is an isomorphism of $\mathcal{O}$-modules. Define $\Phi(f)$ to be the complex which is $e^n\mathcal{O}$ in degree n and has trivial differentials. Then Φ is a homomorphism from $\Gamma(C(\operatorname{spec}\mathcal{O}))$ to $\mathrm{Pic}(\mathcal{D}_{\mathcal{M}})$ such that $\tilde{\Psi} \circ \Phi$ is the identity on $\Gamma(C(\operatorname{spec}\mathcal{O}))$. The naturality is easily verified. □

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