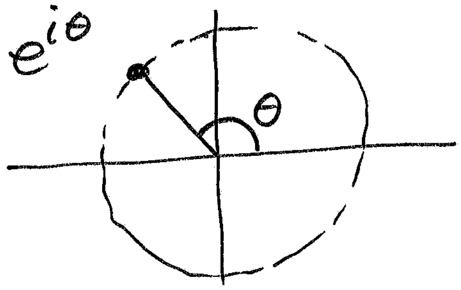
13 jan 2016

Eulers formel

(1) $$e^{i\theta} = \cos\theta + i\sin\theta$$



lengde

vinkel.

$$z = r e^{i\theta}$$

polar form

(skrivemåte for komplekse tall på polar form.)

$$3 \cdot i = 3 e^{i\pi/2}$$ (lengde 3, vinkel $\frac{\pi}{2}$ rad)

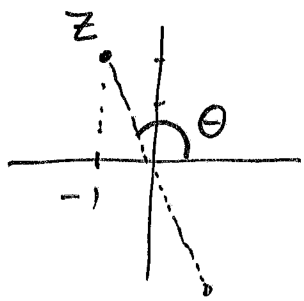
$$e^{2 - i\pi/3} = e^{2} \cdot e^{-i\pi/3}$$

$$-1 = 1 \cdot e^{\pi i} = e^{\pi i}.$$

$$-2\, e^{3\pi i/4}$$ (ikkje på polar form)

$$= 2 e^{3\pi i/4} \cdot e^{\pi i} = 2 e^{7\pi i/4}$$ (på polar form)

$z = -1 + 2i$ på polar form:



$$r = |z| = \sqrt{1^2 + 2^2} = \sqrt{5}$$

$$\tan\theta = \frac{2}{-1} = -2$$

$$\theta = \arctan(-2) = -1{,}107 \text{ rad}$$ $(\sim -63{,}4°)$

$x = -1 < 0$ så vinkelen er $\theta = \arctan(-2) + \pi$

$= 2.034$

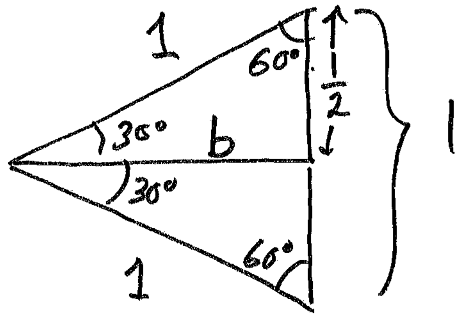
$$z \cong \sqrt{5}\, e^{2.034 i} \sim \underline{2.236 \cdot e^{2.034 i}}$$

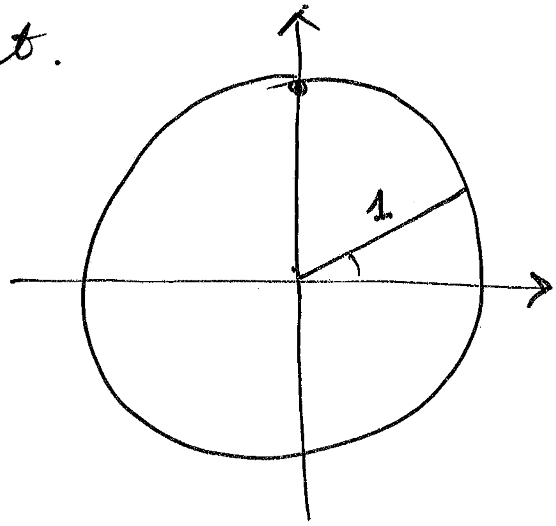
$z = 1 + \sqrt{3}\,i \qquad = 2\,e^{\pi i/3}$

②

	$0 = 0°$	$\frac{\pi}{6} = 30°$	$\frac{\pi}{4} = 45°$	$\frac{\pi}{3} = 60°$	$\frac{\pi}{2} = 90°$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

Lær gjerne disse verdiene utenat.





Pytagoras: $b^2 + \left(\frac{1}{2}\right)^2 = 1^2$

$$b^2 = 1 - \frac{1}{4} = \frac{3}{4}$$

$$b = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{\sqrt{4}} = \frac{\sqrt{3}}{2} \sim 0.866$$

(Tilfellet med 45° er vist i notatene 11.01.2016)

Logaritmer

$$e^{\ln x} = x \qquad x > 0$$

$$10^{\log x} = x$$

$\text{Log } 1000$

$\text{Log}\,(10^3) = 3$

$$\ln(1) = 0$$

$$\ln(a \cdot b) = \ln a + \ln b$$

$$\ln(a^r) = r \ln(a).$$

$$e^z = 1$$

$z = 0 \qquad e^0 = 1$

$z = 2\pi i \qquad e^{2\pi i} = 1$

$$e^{\text{Log } z} = z$$

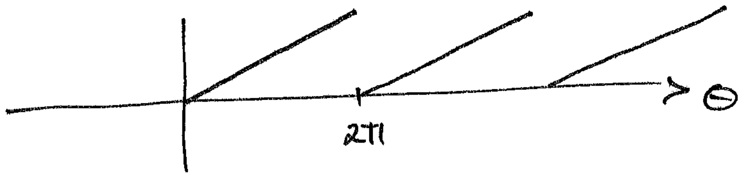
z komplekst tall.

Log z er ikke unikt definert!

Må gjøre valg i hvilke imaginærdeler Log z skal gi.

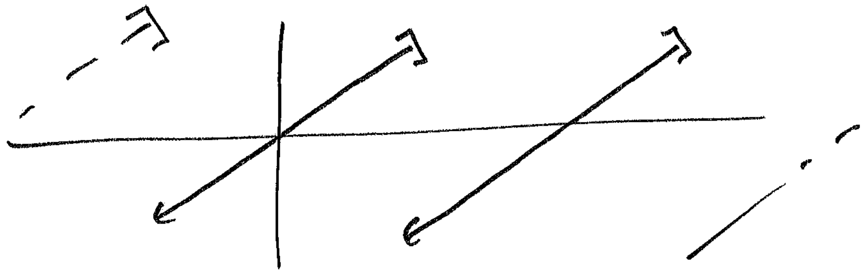
Anta imaginærdel er i intervallet $[0, 2\pi\rangle$.

$\text{Log}\,(e^{i\theta})$



Anta imaginærdelen er i intervallet $\langle -\pi, \pi]$

$\mathrm{Log}(e^{i\theta})$



(4)

(dette gjøres i Matlab)

$$\mathrm{Log}(2e^{\pi i/4}) = \mathrm{Log}(e^{\ln 2} \cdot e^{\pi i/4})$$
$$= \mathrm{Log}(e^{\ln 2 + \pi i/4})$$
$$= \ln 2 + \pi i/4$$

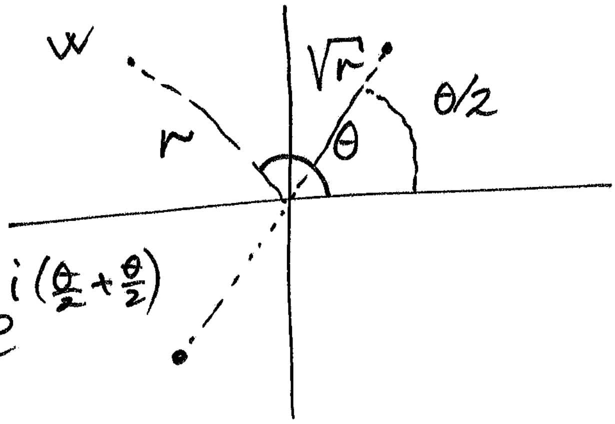
$$\mathrm{Log}(r e^{i\theta}) = \ln(r) + 2\pi i \cdot m + i\theta$$

for en passende $m \in \mathbb{N}$ naturlig tall.

⑤

Kvadratrøtter

Kvadratrøttene til w er løsningene til likningen $z^2 = w$.



$w = re^{i\theta}$

$$\left(\sqrt{r}\,e^{i\theta/2}\right)^2 = \left(\sqrt{r}\right)^2 \cdot e^{i\left(\frac{\theta}{2}+\frac{\theta}{2}\right)}$$
$$= w$$

så $\sqrt{r}\,e^{i\theta/2}$ er en kvadratrot.

$$\left(-\sqrt{r}\,e^{i\theta/2}\right)^2 = (-1)^2\left(\sqrt{r}\,e^{i\theta/2}\right)^2 = w.$$

Kvadratrøttene er $\underline{\pm\sqrt{r}\,e^{i\theta/2}}$

Eksempel Kvadratrøttene til i.

$$i = 1\cdot e^{\pi i/2} = e^{\pi i/2}$$

Røttene er $\pm\sqrt{1}\,e^{(\pi i/2)/2}$

$$\pm e^{\pi i/4}$$

$$\underline{\pm\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\,i\right)}$$

Vi må velge →

$\sqrt{i}$ avhenger av valg. intervall for vinkler som benyttes når tallene skrives på polar form.

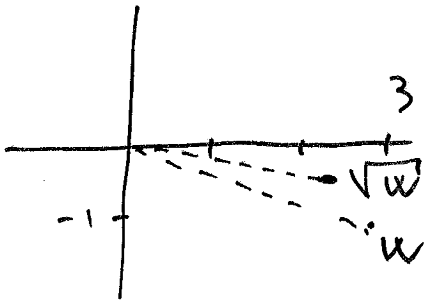
Eksempel $\sqrt{3-i}$, $w = 3-i$ (1f i boka)

$$|w| = \sqrt{3^2+(-1)^2} = \sqrt{10}$$

(6) $\arctan\left(\frac{-1}{3}\right) = \arctan\left(\frac{-1}{3}\right) = -0.3217\ldots$

x-komponenten er positiv.

$$w = \sqrt{10}\, e^{-0.3217 \cdot i}$$



$$\sqrt{w} = \sqrt{\sqrt{10}} \cdot e^{-0.3217 i/2}$$

$$\sim \underline{\sqrt[4]{10}\, e^{-0.1608 i}}$$

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n-røttene til w er løsningene

til $z^n = w$ $(n = 2, 3, \ldots)$

(n løsninger når $w \neq 0$)

$$w = s e^{i \cdot v}$$

s lengde
v vinkel

$$z = r e^{i\theta}$$

$$z^n = r^n e^{i \cdot n\theta} = w = s e^{i \cdot v}$$

(7)

$$r^n = s$$
$$n \cdot \theta = v + 2\pi \cdot m$$

(m antall omløp)

$$\underline{r = \sqrt[n]{s}}$$
$$\underline{\theta = \frac{v}{n} + \frac{2\pi}{n} \cdot m}$$

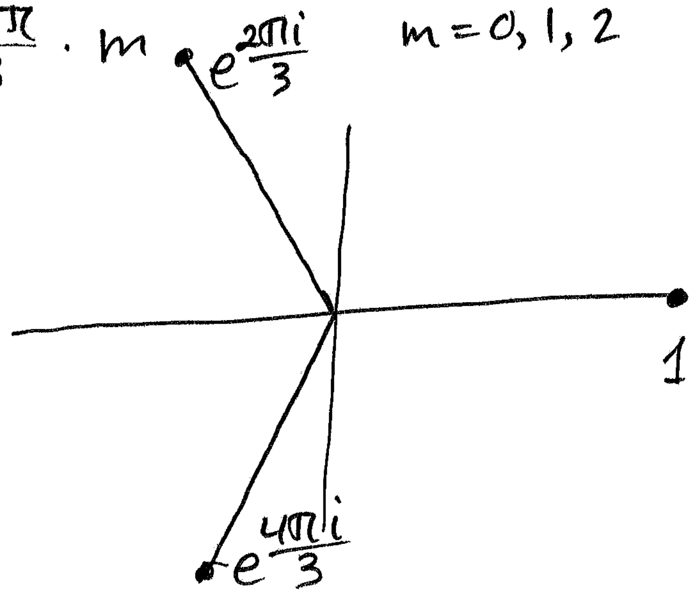
3-røttene til 1.

$$1 = 1 \cdot e^{0 \cdot i}$$

$s = 1$
$v = 0$

Løsningene er $z = r e^{i\theta}$

$$r = \sqrt[3]{1} = 1$$
$$\theta = \frac{0}{3} + \frac{2\pi}{3} \cdot m \qquad m = 0, 1, 2$$

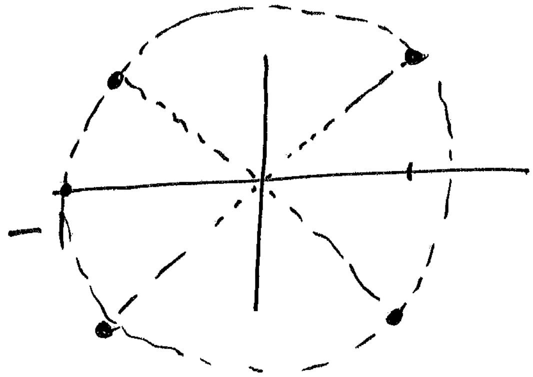


$$z = 1 e^{0 \cdot i} = 1$$
$$z = 1 e^{2\pi i/3}$$
$$z = 1 \cdot e^{4\pi i/3}$$

$$z^3 - 1 = (z - 1)\underbrace{(z - e^{2\pi i/3})(z - e^{4\pi i/3})}_{z^2 + z + 1}$$

(8)

$$z^4 = -1$$
$$= 1 \cdot e^{\pi i}$$



$$z = r e^{i\theta}$$

$$z^4 = r^4 e^{i \cdot 4\theta} = 1 \cdot e^{\pi \cdot i + 2\pi i \cdot m}$$

$$r^4 = 1 \qquad \underline{r = 1}$$

$$4\theta = \pi + 2\pi \cdot m$$
$$\theta = \frac{\pi}{4} + \frac{\pi}{2} \cdot m$$

Røttene er $\underline{e^{\pi i/4},\ e^{3\pi i/4},\ e^{-\pi i/4},\ e^{-3\pi i/4}}$

$\left(\overset{\shortparallel}{e^{7\pi i/4}}\right)$ $\left(\overset{\shortparallel}{e^{5\pi i/4}}\right)$

$z^4 + 1$ har røttene ovenfor.

$$z^4 + 1 = \left(z - e^{\pi i/4}\right)\left(z - e^{-\pi i/4}\right)\left(z - e^{3\pi i/4}\right) \cdot$$
$$\left(z - e^{-3\pi i/4}\right)$$

$$= \underline{\left(z^2 - \sqrt{2}\, z + 1\right)\left(z^2 + \sqrt{2}\, z + 1\right)}$$

Test 13.01.2016

1. Gitt $z = 3-4i$

Bestem a) $\mathrm{Re}(z)=3$ og $\mathrm{Im}(z)=-4$

$\bar{z} = \sqrt{3^2+(-4)^2} = \sqrt{25} = 5$

b) $\bar{z} = 3+4i$ (kompleks konjugerte til z)

c) $|z|=5$ (lengden til z)

$\frac{1}{z} = \frac{\bar{z}}{|z|^2} \left(= \frac{\bar{z}}{z \cdot \bar{z}} \right)$

d) $\frac{1}{z} = \frac{3+4i}{5^2} = \frac{3+4i}{25}$ (på kartesisk form)

e) z^2

$= (3-4i)^2 = 3^2 + \underbrace{(-4i)^2}_{(-1)^2 \cdot 4^2 \cdot i^2} + 2 \cdot 3(-4i)$

$= 9 - 16 + (-24i)$

$= -7 - 24i$

bensttter ($i^2=-1$)

2. Finn den eksakte løsningen til likningen

$$i + (1+\sqrt{3}\,i)\, z = 1$$

Skriv løsningen på formen $re^{i\theta}$

(Eksamens oppgave Vår 2015)