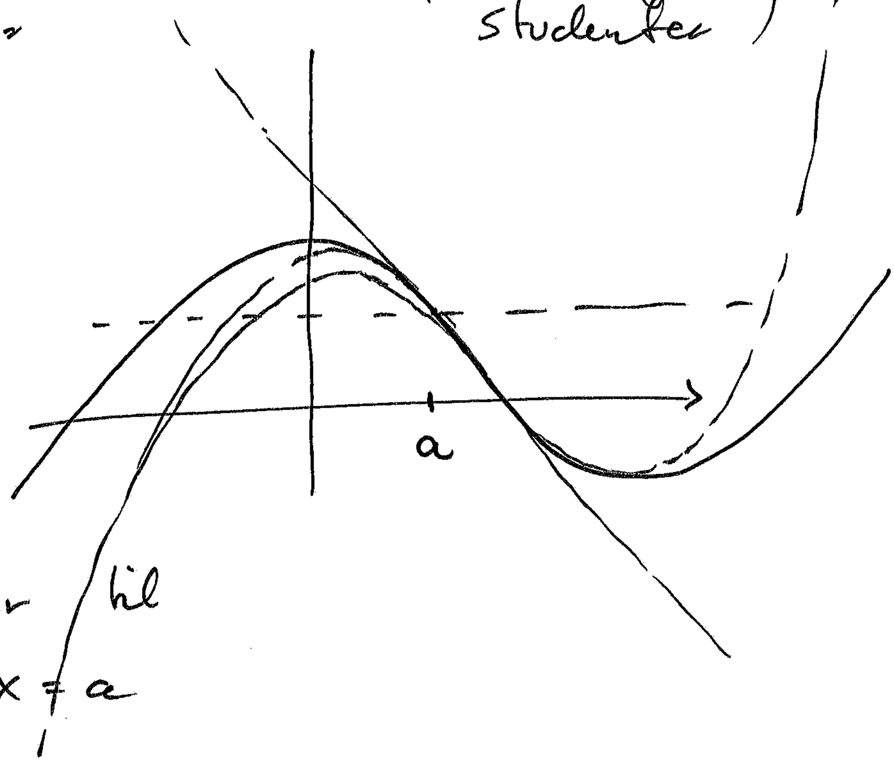
Taylor polynomer

26 nov. 2015

(spørsmål fra studenter)

$f(x)$ "glatt"

(1)



Taylor polynomer til $f(x)$ rundt $x = a$

$$T_0(x) = f(a)$$

$$T_1(x) = f(a) + f'(a)(x-a)$$

$$T_2(x) \overset{\text{her}}{=} T_1(x) \qquad (a \text{ et vendepunkt?})$$

$$T_2(x) \overset{\text{generelt:}}{=} f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2$$

$$T_3(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2 + \frac{1}{6} f^{(3)}(a)(x-a)^3$$

etc

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!} (x-a)^i$$

De deriverte til $f(x)$ og $T_n(x)$ i $x = a$ er like opp til og med n-te deriverte.

Eksempel:

$f(x) = \ln(x^2-3)$ om $x = 2$.

(2) Finn $T_2(x)$ om 2.

$$T_0(x) = f(2) = \ln(2^2-3) = \ln 1 = 0$$

$$T_1(x) = f(2) + f'(2)(x-2)$$

$$f'(x) = \frac{d\ln(u)}{du} \cdot \frac{du}{dx} \quad \text{kjerneregel} \quad u = x^2-3$$

$$= \frac{1}{x^2-3} \cdot (x^2-3)' = \frac{2x}{x^2-3}$$

$$f'(2) = \frac{2 \cdot 2}{1} = 2 \cdot 2 = 4$$

$$T_1(x) = 0 + 4(x-2)$$

$$f''(x) = (f'(x))' = \left(\frac{2x}{x^2-3}\right)' = \left(2x \cdot \frac{1}{x^2-3}\right)' \quad \text{(produkt regel)}$$

$$= 2 \cdot \frac{1}{x^2-3} + 2x\left(\frac{-1}{(x^2-3)^2} \cdot (x^2-3)'\right)$$

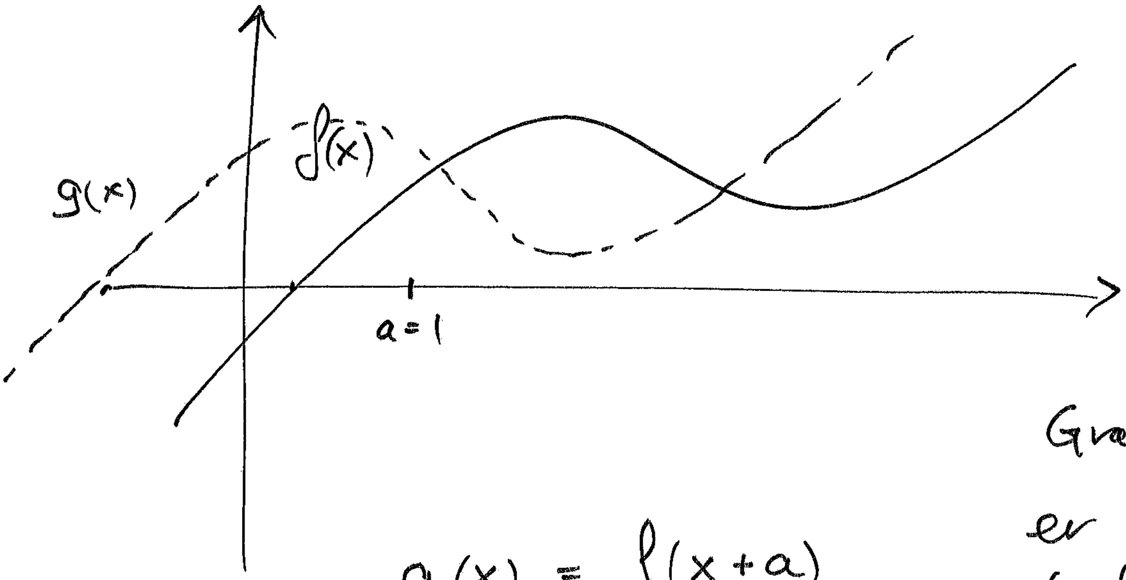
$$= \frac{2}{x^2-3} - \frac{(2x)^2}{(x^2-3)^2}.$$

$$f''(2) = \frac{2}{1} - \frac{(2\cdot 2)^2}{1^2} = 2 - 16 = -14$$

$$T_2(x) = 4(x-2) + \frac{1}{2}f''(2)(x-2)^2$$

$$= 4(x-2) + \frac{-14}{2}(x-2)^2$$

$$= \underline{4(x-2) - 7(x-2)^2}$$



Grafen til g er lik grafen til f forskyvd med $-a$

③

$$g(x) = f(x+a)$$

$$g(x-a) = f(x)$$

$$f(x) = \frac{1}{1-x}$$

$$T_n(x) = 1 + x + x^2 + \cdots + x^n \qquad |x| < 1$$

$$f(x) - T_n(x) = x^{n+1}\frac{1}{1-x}$$

$(x \neq 1)$

restleddet lite for $|x|<1$ og n stor.

$\int \frac{1}{x}\,dx = \ln|x| + c$ (husk $|x|$ ...)

$\ln(x^2+1) = \ln|x^2+1|$ def. for alle x.

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$\int \frac{1}{2x-5}\,dx =$

$u = 2x-5$
$du = 2dx$
$\frac{1}{2}du = dx$

$\int \frac{1}{u} \cdot \frac{1}{2}du$ lett å glemme

$= \frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\ln|u| + c$

$= \underline{\frac{1}{2}\ln|2x-5| + c}$

$$\int x^r\,dx = \begin{cases} \frac{x^{r+1}}{r+1} + c & r \neq -1 \\ \ln|x| + c \end{cases}$$

$\int \frac{1}{\sqrt{3x-5}}\,dx$

$u = 3x-5$
$du = 3dx$
$\frac{1}{3}du = dx$

$= \int \frac{1}{\sqrt{u}}\,\frac{1}{3}du$

$= \frac{1}{3}\int u^{-1/2}\,du = \frac{1}{3}\cdot\frac{u^{1/2}}{1/2} + c$

$= \underline{\frac{2}{3}\sqrt{3x-5} + c}$

$$u = x^2 - 1$$
$$du = 2x\,dx$$
$$\tfrac{1}{2}du = x\,dx$$

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$$\int x \cdot \tan(x^2 - 1)\,dx$$

$$= \int \tan(u)\tfrac{1}{2}du$$

$$= \frac{1}{2}\int \frac{\sin(u)}{\cos(u)}\,du$$

$$V = \cos u$$
$$dV = -\sin u\,du$$

$$= \frac{1}{2}\int \frac{-1}{V}\,dV$$

$$= -\frac{1}{2}\ln|V| + c$$

$$= -\frac{1}{2}\ln|\cos(u)| + c$$

$$= \underline{-\frac{1}{2}\ln|\cos(x^2-1)| + c}$$

Finn arealet A avgrenset av x-aksen, grafen til x^2-1, mellom $x=0$ og $X=2$, og linjene $x=0$ og $x=2$.

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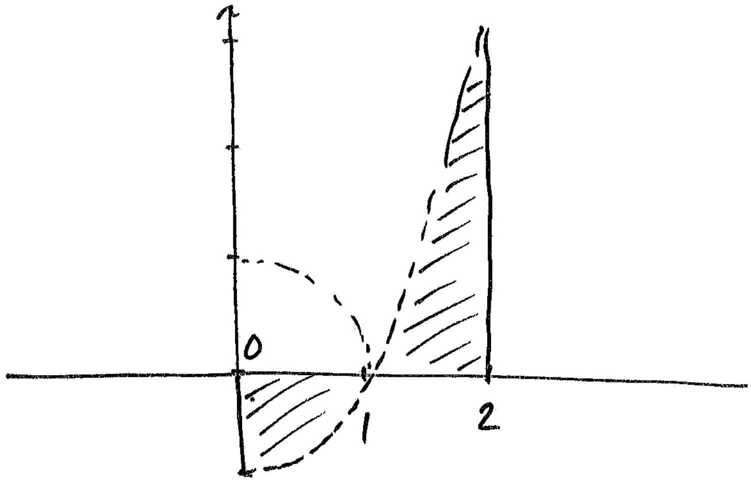

$$A = \int_0^2 |x^2-1|\,dx = -\int_0^1 x^2-1\,dx + \int_1^2 x^2-1\,dx$$

$$= -\left(\frac{x^3}{3}-x\right)\Big|_0^1 + \left(\frac{x^3}{3}-x\right)\Big|_1^2$$

$$= -\left(\frac{1}{3}-1\right) + \frac{8-1}{3}-(2-1) \quad \left(\frac{8}{3}-2-\left(\frac{1}{3}-1\right) = \frac{8}{3}-\frac{1}{3}-(2-1)\right)$$

$$= \frac{2}{3} + \left(\frac{7}{3}-\frac{3}{3}\right)$$

$$= \frac{2}{3} + \frac{4}{3} = \frac{6}{3} = \underline{\underline{2}}$$

Uegentlig integral

$$\int_1^\infty \frac{1}{x^r}\,dx$$

Når eksisterer integralet?

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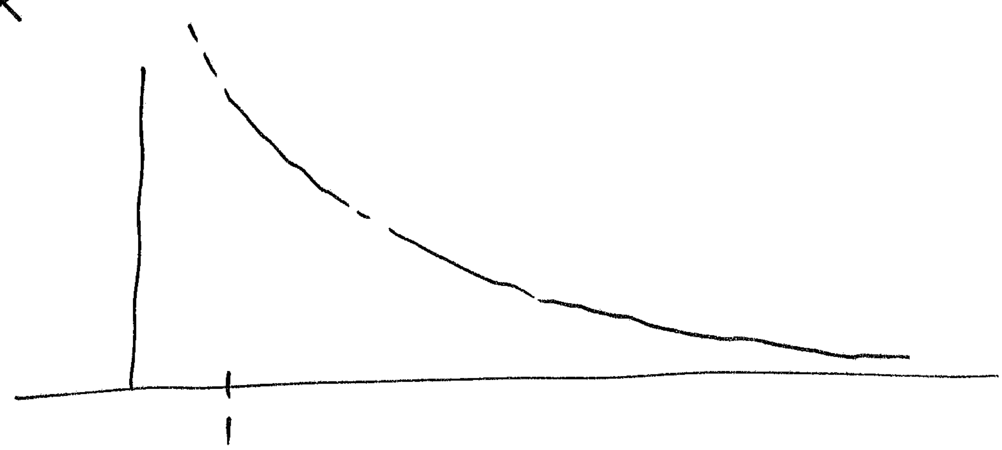

$$\lim_{N\to\infty} \int_1^N \frac{1}{x^r}\,dx$$

$r=1$ $\quad \lim_{N\to\infty} \ln|x| \Big|_1^N = \lim_{N\to\infty} \ln(N) = \infty$

integralet eksisterer ikkje.

$r\neq 1$ $\quad \lim_{N\to\infty} \int_1^N x^{-r}\,dx = \lim_{N\to\infty} \frac{x^{-r+1}}{1-r}\Big|_1^N$

$$= \lim_{N\to\infty} \frac{N^{1-r}-1}{1-r}.$$

(eks $r=2$ $\quad \lim_{N\to\infty} \frac{1/N - 1}{-1} = 1$)

$r>1$ $\quad \int_1^\infty \frac{1}{x^r}\,dx = \underline{\underline{\frac{1}{r-1}}}$ (eksisterer)

$r<1$ $\quad$ integralet eksisterer ikkje.

$$\int_0^1 \frac{1}{x^r}\,dx$$ For hvilke r eksisterer integralet?

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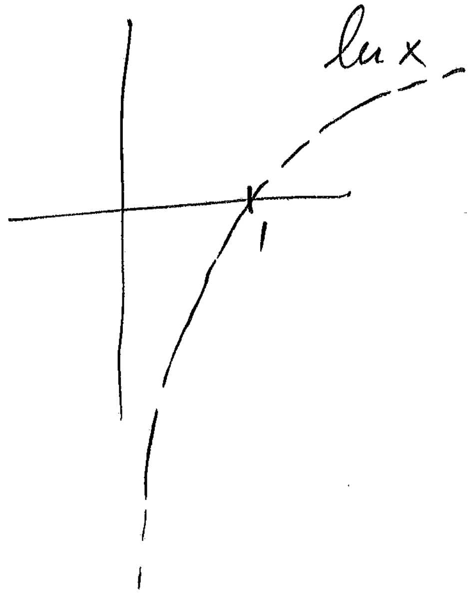
$$\lim_{d\to 0^+} \int_d^1 \frac{1}{x^r}\,dx = \lim_{d\to 0^+} \int_d^1 x^{-r}\,dx$$

$$= \lim_{d\to 0^+} \begin{cases} \left.\dfrac{x^{-r+1}}{1-r}\right|_d^1 & r \neq 1 \\ \ln|x| \,\Big|_d^1 & r = 1 \end{cases}$$



$$= \lim_{d\to 0^+} \begin{cases} \dfrac{1 - d^{-r+1}}{1-r} & r \neq 1 \\ -\ln d & r = 1 \end{cases}$$

$r=1$ $\int_0^1 \frac{1}{x}\,dx$ eksisterer ikkje.

$r>1$ $\int_0^1 \frac{1}{x^r}\,dx$ —— " ——

$r<1$ $\int_0^1 \frac{1}{x^r}\,dx = \frac{1}{1-r}$ (eksisterer) som uegentlig integral

$$\lim_{x\to\infty} e^{-x} x^2 = \lim_{x\to\infty} \frac{x^2}{e^x} = 0$$

L'Hopitals regel $\left(\frac{\infty}{\infty}\right)$

$$\lim_{x\to\infty} \frac{2x}{e^x} \qquad \left(\frac{\infty}{\infty}\right) \quad \text{L'Hopitals regel}$$

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$$\lim_{x\to\infty} \frac{2}{e^x} = 0$$

Tilsvarende for alle polynomer $p(x)$

$$\lim_{x\to\infty} \frac{p(x)}{e^x} = 0$$

"Eksponentiell funksjoner vokser raskere enn polynom funksjoner."

Vekselstrøm

$U(t) = V_0 \sin(\omega t)$ (når $U(0) = 0$)

Frekvens 50 HZ $\omega = 2\pi \cdot 50$

Spenning 220 V Hva er V_0?

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$R \cdot I = U$ ohms lov

Effekt $P = I \cdot U = \frac{U^2}{R}$

Gjennomsnittseffekt $= \frac{1}{R}$ (gjennomsnitt til U^2)

Hva er gjennomsnittsspenningen?

$\sqrt{\text{gjennomsnitt til } U^2} = U_{rms}$

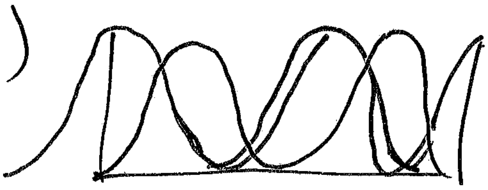
$\frac{1}{T}\int_0^T U^2\, dt = \frac{1}{T}\int V_0^2 \sin^2(\omega t)\, dt$

T (slik at vi får et helt antall perioder)

$\frac{1}{2\pi}\int_0^{2\pi} \sin^2(t)\, dt = \frac{1}{2}$

(Antiderivert: $\frac{1}{2}(1-\cos(2t)) = \sin^2 t$ etc)

$\int_0^{2\pi} \sin^2 t\, dt \overset{\text{Like}}{=} \int_0^{2\pi} \cos^2 t\, dt$



$= \frac{1}{2}\int_0^{2\pi} \underbrace{\sin^2 t + \cos^2 t}_{1}\, dt = \underline{\underline{\pi}}$

"triks for å finne integralene"

$$U_{rms} = \sqrt{\frac{1}{2} \cdot V_0^2} = \frac{V_0}{\sqrt{2}}$$

$$V_0 = \sqrt{2} \cdot U_{rms}$$

$$\sqrt{2} \cdot 220\,V \approx \underline{\underline{311\,V}}$$

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