LF MA1000, 7. juni 2010

1.

a)

$$\left(\begin{array}{ccc|c} 0 & 1 & 3 & 4 \\ -3 & 0 & 2 & -2 \\ 1 & 1 & 2 & 3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ -3 & 0 & 2 & -2 \\ 0 & 1 & 3 & 4 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & 3 & 8 & 7 \\ 0 & 1 & 3 & 4 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 3 & 8 & 7 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & -1 & -5 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 5 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 0 & -7 \\ 0 & 1 & 0 & -11 \\ 0 & 0 & 1 & 5 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -11 \\ 0 & 0 & 1 & 5 \end{array}\right)$$

$\underline{\underline{x = 4,\ y = -11,\ z = 5}}$

b)

$$\begin{vmatrix} a+1 & 1 & 3 \\ -3 & a+1 & a+3 \\ 1 & 1 & a+3 \end{vmatrix} = (a+1)\cdot\begin{vmatrix} a+1 & a+3 \\ 1 & a+3 \end{vmatrix} - (-3)\begin{vmatrix} 1 & 3 \\ 1 & a+3 \end{vmatrix} + 1\cdot\begin{vmatrix} 1 & 3 \\ a+1 & a+3 \end{vmatrix}$$

$$= (a+1)\big(a\cdot(a+3)\big) + 3\cdot a + (a+3-3a-3)$$

$$= a\cdot\big((a+1)(a+3)+1\big) = a\left(a^2+4a+4\right) = a\cdot(a+2)^2$$

Ikke-trivielle løsn $\Leftrightarrow \begin{vmatrix} \vdots & \vdots & \vdots \end{vmatrix} = 0 \Leftrightarrow \underline{\underline{a=0}}$ eller $\underline{\underline{a=-2}}$

$\underline{a=0}$:

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ -3 & 1 & 3 & 0 \\ 1 & 1 & 3 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 4 & 12 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left.\begin{array}{l} x=0 \\ y=-3z \\ z \text{ fri} \end{array}\right\} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = z\cdot\underline{\underline{\begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix}}} \Rightarrow \text{Basis: } \underline{\underline{\left\{\begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix}\right\}}}$$

$\underline{a=-2}$:

$$\left(\begin{array}{ccc|c} -1 & 1 & 3 & 0 \\ -3 & -1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} -1 & 1 & 3 & 0 \\ 0 & -4 & -8 & 0 \\ 0 & 2 & 4 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left.\begin{array}{l} x=z \\ y=-2z \\ z \text{ fri} \end{array}\right\} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = z\cdot\underline{\underline{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}}} \Rightarrow \text{Basis: } \underline{\underline{\left\{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}\right\}}}$$

c) $\left(\begin{array}{ccc|c} a+1 & 1 & 3 & 3-a \\ -3 & a+1 & a+3 & a-1 \\ 1 & 1 & a+3 & 3 \end{array}\right)$ For $a \neq 0, -2$ så er koeff. matrixen inverterbar $\Rightarrow$ én løsning $\Rightarrow$ konsistent

$\underline{a=0}$: $\left(\begin{array}{ccc|c} 1 & 1 & 3 & 3 \\ -3 & 1 & 3 & -1 \\ 1 & 1 & 3 & 3 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 3 & 3 \\ 0 & 4 & 12 & 8 \\ 0 & 0 & 0 & 0 \end{array}\right)$ konsistent, én fri variabel

$\underline{a=-2}$: $\left(\begin{array}{ccc|c} -1 & 1 & 3 & 5 \\ -3 & -1 & 1 & -3 \\ 1 & 1 & 1 & 3 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} -1 & 1 & 3 & 5 \\ 0 & -4 & -8 & -18 \\ 0 & 2 & 4 & 8 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} -1 & 1 & 3 & 5 \\ 0 & 2 & 4 & 8 \\ 0 & 0 & 0 & -2 \end{array}\right)$ inkonsistent

Inkonsistent: $\underline{\underline{a=-2}}$

2. a) $f(x) = \ln(x^2-2x+3) \Rightarrow f'(x) = \frac{1}{x^2-2x+3} \cdot (2x-2) = \underline{\underline{\frac{2x-2}{x^2-2x+3}}}$

$g(x) = x^2 \cdot \sin(3-x) \Rightarrow$

$g'(x) = x^2 \cdot \cos(3-x) \cdot (-1) + 2x \sin(3-x)$

$= \underline{\underline{2x \sin(3-x) - x^2 \cos(3-x)}}$

b) $x^2 y^3 = y \ln x + x^3$

$2x \cdot y^3 + x^2 \cdot 3y^2 \cdot y' = y' \cdot \ln x + y \cdot \frac{1}{x} + 3x^2$

$y' \cdot (3x^2y^2 - \ln x) = 3x^2 + \frac{y}{x} - 2xy^3$

$(x,y) = (1,1)$: $y' \cdot 3 = 2 \Rightarrow y' = \underline{\underline{2/3}}$

3.

a) $\int x^2 e^x\,dx = x^2 e^x - \int 2x \cdot e^x\,dx$

$= x^2 e^x - 2\left(xe^x - \int 1 \cdot e^x\,dx\right)$

$= x^2 e^x - 2xe^x + 2e^x + C = \underline{(x^2-2x+2)e^x + C}$

$\int \sqrt{x}\, e^{\sqrt{x}}\,dx = \int \sqrt{x}\, e^u\, 2\sqrt{x}\,du = 2\int u^2 e^u\,du$

$\boxed{u=\sqrt{x} \quad du = \frac{1}{2\sqrt{x}}\,dx}$

$= 2\left((\sqrt{x})^2 - 2\sqrt{x}+2\right)e^{\sqrt{x}} + C$

$= \underline{\underline{(2x - 4\sqrt{x}+4)e^{\sqrt{x}} + C}}$

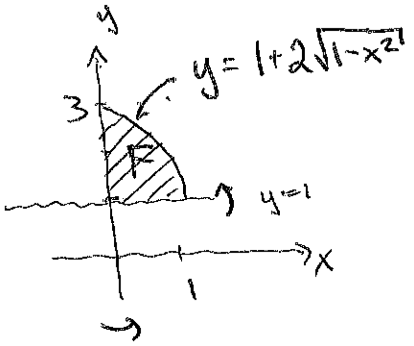
b) $\int_0^{\pi/2} \frac{4\cos x}{1+\sin^2 x}\,dx = \int_0^1 \frac{4\,du}{1+u^2} = 4\arctan(u)\Big|_0^1$

$\boxed{u=\sin x \quad du = \cos x\,dx \quad u(0)=0 \quad u(\pi/2)=1}$

$= 4\arctan(1) - 4\arctan(0)$

$= 4 \cdot \pi/4 = \underline{\underline{\pi}}$

c)



(1) Rotasjon om y-aksen:

$V = \int_0^1 2\pi x \cdot (y-1)\,dx = \int_0^1 2\pi x \cdot 2\sqrt{1-x^2}\,dx$

$= 2\pi\left[-\frac{2}{3}(1-x^2)^{3/2}\right]_0^1 = 2\pi\left(0+\frac{2}{3}\right) = \underline{\underline{\frac{4\pi}{3}}}$

(2) Rotasjon om $y=1$:

$V = \int_0^1 \pi \cdot (y-1)^2\,dx = \int_0^1 \pi \cdot 4(1-x^2)\,dx$

$= 4\pi\left[x - \frac{1}{3}x^3\right]_0^1 = 4\pi \cdot \frac{2}{3} = \underline{\underline{\frac{8\pi}{3}}}$

4.

a) $xy' + y = 8x^3 - 4x$

$(xy)' = 8x^3 - 4x$

$x \cdot y = \int 8x^3 - 4x \, dx = 2x^4 - 2x^2 + C$

$y = \frac{2x^4 \mp 2x^2 + C}{x} = 2x^3 \mp 2x + \frac{C}{x}$

$\underline{y(1)=0}$: $0 = 2 \cdot 1 \mp 2 \cdot 1 + \frac{C}{1} = C \Rightarrow \underline{C = 0}$

$y = \underline{\underline{2x^3 - 2x}}$

b) $y'' - 4y' + 5y = 10$:

Homogen løsning y_h: $y'' - 4y' + 5y = 0$

Karakteristisk likn: $r^2 - 4r + 5 = 0 \Rightarrow r = \frac{4 \pm \sqrt{16 - 20}}{2} = 2 \pm i$

$y_h = \underline{e^{2x}(C_1 \cos x + C_2 \sin x)}$

Partikulær løsn y_p: Prøver $y_p = A$ (konstant)

$y_p' = 0$

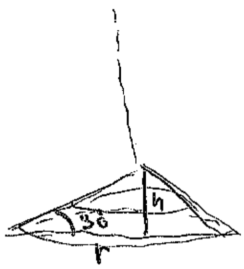
$y_p'' = 0$

$5y_p = 5A = 10 \Rightarrow A = 2 \Rightarrow y_p = \underline{2}$

Generell løsning: $y = y_h + y_p = \underline{\underline{e^{2x} \cdot (C_1 \cos x + C_2 \sin x) + 2}}$

5.

$\downarrow v = 0.30\,m^3/s$



$\frac{h}{r} = \tan 30^\circ = \frac{1}{\sqrt{3}}$

$\Downarrow$

$h = r/\sqrt{3}$

a) $V = \pi r^2 \cdot h/3 = \pi \cdot (\sqrt{3}h)^2 \cdot h/3 = \pi \cdot h^3$

$$\left.\begin{aligned} \frac{dV}{dh} &= 3\pi h^2 \\ \frac{dV}{dt} &= 0.30 \end{aligned}\right\} \quad \frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$$

$\Downarrow$

$$h' = \frac{dh}{dt} = \frac{0.30}{3\pi h^2} = \frac{0.30}{3\pi} \cdot \frac{1}{1.0^2}$$

$$= 0.1 \cdot \frac{1}{\pi} \simeq 0.0318\ m/s$$

$$= 3.2\ cm/s$$

b) $h' = \frac{0.30}{3\pi \cdot h^2} \Rightarrow$

$t=0, h=0 \Rightarrow$

$$\boxed{\begin{aligned} 3h^2 \cdot h' &= \frac{1}{10\pi} \\ h(0) &= 0 \end{aligned}}$$

$$\int 3h^2 \cdot h' \, dt = \int \frac{1}{10\pi} \, dt$$

$$h^3 = \frac{t}{10\pi} + C$$

$h(0)=0: \quad 0^3 = 0 + C \Rightarrow C = 0$

$h^3 = \frac{t}{10\pi}$

$h = 3{,}0: \quad (3{,}0)^3 = \frac{t}{10\pi} \Rightarrow t = 10\pi \cdot 9 = 90\pi$

$\simeq 282\ s$

$(90 \cdot 3.14 \simeq 282)$

$= 4\ min\ 42\ s$

6.

a) $\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 2 \\ -1 & 2 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 2 & 1-\lambda \end{vmatrix} - 1\cdot\begin{vmatrix} 1 & 1 \\ 1-\lambda & 2 \end{vmatrix}$

$= (1-\lambda)\cdot\left((1-\lambda)^2-4\right) - 1\left(\cancel{(1-\lambda)-2}\right) - 1\left(\cancel{2-(1-\lambda)}\right)$

$= (1-\lambda)\,(\lambda^2-2\lambda-3) = -(\lambda-1)(\lambda-3)(\lambda+1) = 0$

Egenverdier: $\underline{\lambda_1 = 1,\ \lambda_2 = 3,\ \lambda_3 = -1}$

b) A er 3×3 med 3 ulike egenverdier $\Rightarrow$ A diagonaliserbar

Egenrom: $\lambda = 1 \Rightarrow \left(\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \\ -1 & 2 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ $\quad x = -2z,\ y = -z,\ z$ fri

$\Rightarrow \underline{v}_1 = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$ Basis for E_1: $\{\underline{v}_1\}$

$\lambda = 3 \Rightarrow \left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 2 & 0 \\ -1 & 2 & -2 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -2 & 2 & 0 \\ 0 & -3 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -4/3 & 0 \\ 0 & 1 & -5/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = z\cdot\begin{pmatrix} 4/3 \\ 5/3 \\ 1 \end{pmatrix} \Rightarrow \underline{v}_2 = \begin{pmatrix} 4 \\ 5 \\ 3 \end{pmatrix}$ Basis for E_3: $\{\underline{v}_2\}$

$\lambda = -1 \Rightarrow \left(\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 1 & 2 & 2 & 0 \\ -1 & 2 & 2 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ $\quad x = 0,\ y = -z,\ z$ fri

$\Rightarrow \underline{v}_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ Basis for E_{-1}: $\{\underline{v}_3\}$

Konklusjon: $A = P\cdot D\cdot P^{-1}$ når $P = \begin{pmatrix} -2 & 4 & 0 \\ -1 & 5 & -1 \\ 1 & 3 & 1 \end{pmatrix}$ og $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

c) Vi har at hvis $\underline{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \underline{v}e^{\lambda t} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} e^{\lambda t}$, så er

$\underline{y}' = A\underline{y} \iff A\underline{v} = \lambda\underline{v}$, dvs λ egenverdi og $\underline{v}$ egenvektor for A.

Dermed er lineær-kombinasjonen

$\underline{y} = C_1\cdot \underline{v}_1 e^{\lambda_1 t} + C_2 \underline{v}_2 e^{\lambda_2 t} + C_3 \underline{v}_3 e^{\lambda_3 t}$

den generelle løsningen av $A \cdot \underline{y} = \underline{y}'$ med $\begin{cases} \lambda_1 = 1 \\ \lambda_2 = 3 \\ \lambda_3 = -1 \end{cases}$ og $\begin{cases} \underline{v_1} \\ \underline{v_2} \\ v_3 \end{cases}$ som ovenfor. Dermed får vi:

$$\underline{y} = C_1 \cdot \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} e^{t} + C_2 \cdot \begin{pmatrix} 4 \\ 5 \\ 3 \end{pmatrix} e^{3t} + C_3 \cdot \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} e^{-t}$$

eller

$$\begin{aligned} y_1 &= -2C_1 e^{t} + 4C_2 e^{3t} \\ y_2 &= -C_1 e^{t} + 5C_2 e^{3t} - C_3 e^{-t} \\ y_3 &= C_1 e^{t} + 3C_2 e^{3t} + C_3 e^{-t} \end{aligned}$$

d) $\left.\begin{aligned} y_1(0) &= 12 \\ y_2(0) &= 4 \\ y_3(0) &= -4 \end{aligned}\right\} \Rightarrow \underline{y}(0) = \begin{pmatrix} 12 \\ 4 \\ -4 \end{pmatrix} = C_1 \cdot \underline{V_1} e^{\lambda_1 \cdot 0} + C_2 \underline{V_2} e^{\lambda_2 \cdot 0} + C_3 \underline{V_3} e^{\lambda_3 \cdot 0}$

$$= C_1 \underline{V_1} + C_2 \underline{V_2} + C_3 \underline{V_3}$$

<u>Dette gir:</u>

$$P \cdot \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} 12 \\ 4 \\ -4 \end{pmatrix} \Rightarrow \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = P^{-1} \begin{pmatrix} 12 \\ 4 \\ -4 \end{pmatrix}$$

Vi kan løse ved å finne P^{-1} eller vha Gauss:

$$\left(P \,\middle|\, \begin{matrix} 12 \\ 4 \\ -4 \end{matrix}\right) = \left(\begin{matrix} -2 & 4 & 0 \\ -1 & 5 & -1 \\ 1 & 3 & 1 \end{matrix} \,\middle|\, \begin{matrix} 12 \\ 4 \\ -4 \end{matrix}\right) \to \left(\begin{matrix} 1 & 3 & 1 \\ -1 & 5 & -1 \\ -2 & 4 & 0 \end{matrix} \,\middle|\, \begin{matrix} -4 \\ 4 \\ 12 \end{matrix}\right) \to \left(\begin{matrix} 1 & 3 & 1 \\ 0 & 8 & 0 \\ 0 & 10 & 2 \end{matrix} \,\middle|\, \begin{matrix} -4 \\ 0 \\ 4 \end{matrix}\right)$$

$$\to \left(\begin{matrix} 1 & 3 & 1 \\ 0 & 1 & 0 \\ 0 & 10 & 2 \end{matrix} \,\middle|\, \begin{matrix} -4 \\ 0 \\ 4 \end{matrix}\right) \to \left(\begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{matrix} \,\middle|\, \begin{matrix} -4 \\ 0 \\ 4 \end{matrix}\right) \to \left(\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \,\middle|\, \begin{matrix} -6 \\ 0 \\ 2 \end{matrix}\right) \quad \begin{aligned} C_1 &= -6 \\ C_2 &= 0 \\ C_3 &= 2 \end{aligned}$$

<u>Løsning:</u> $\underline{y} = -6 \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} e^{t} + 0 \cdot \begin{pmatrix} 4 \\ 5 \\ 3 \end{pmatrix} e^{3t} + 2 \cdot \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} e^{-t}$

$$= \begin{pmatrix} 12 \\ 6 \\ -6 \end{pmatrix} e^{t} + \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix} e^{-t} = \begin{pmatrix} 12e^{t} \\ 6e^{t} - 2e^{-t} \\ -6e^{t} + 2e^{-t} \end{pmatrix}$$