17.10.22 Kap 7

Høyere ordens deriverte.

$f(x)$ funksjon Deriverer $f'(x)$ (ny) funksjon

D_f — $D_{f'} \subset D_f$ tar vekk x-verdier hvor f ikke er deriverbar

$f = 2x^3 + 4$ $D_f = \mathbb{R}$ — $f' = 6x^2$ $D_{f'} = \mathbb{R}$

Deriverer f' gir $f'' = (f')' = (6x^2)' = 6(x^2)' = 12x$

$f''' = (f'')' = (12x)' = 12$

$f^{(4)}(x) = (f''')' = (12)' = 0$

Dobbel derivert $f^{(2)}(x) = f''(x)$

Trippel derivert $f^{(3)}(x) = f'''(x)$

n-te derivert $f^{(n)}(x)$

$p(x)$ polynom av grad n

$p^{(m)}(x) =$ polynom av grad $n-m$ $m \leq n$

$= 0$ $m > n$.

opg.

1) $f(x) = 5x^4 - 2x^2 - 3x$

$f'(x) = 20x^3 - 4x - 3$

$f''(x) = 60x^2 - 4$

2) $g(x) = \frac{4}{x} + \sqrt{2x+1}$

$= 4 \cdot x^{-1} + (2x+1)^{1/2}$

$g'(x) = -4x^{-2} + \frac{1}{2}(2x+1)^{-1/2} \cdot 2$

$= -4x^{-2} + (2x+1)^{-1/2}$

$g''(x) = (-4)(-2)x^{-3} + \left(\frac{-1}{2}\right)(2x+1)^{-3/2} \cdot 2$

$= 8x^{-3} - (2x+1)^{-3/2}$

$= \frac{8}{x^3} - \frac{1}{\sqrt{(2x+1)^3}}$

derivere flere ganger:

$$x^n, \quad n x^{n-1}, \quad n(n-1)x^{n-2}, \quad n(n-1)(n-2)x^{n-3}, \ldots$$

$$(x^n)^{(k)} = n \cdot (n-1) \cdots (n-k+1)\, x^{n-k}$$

$$(x^n)^{(n)} = n(n-1)(n-2) \cdots 2 \cdot 1 = n!$$ n fakultet.

$1! = 1$, $2! = 2$, $3! = 6$, $4! = 24$, $5! = 120$
$6! = 720 \ldots$

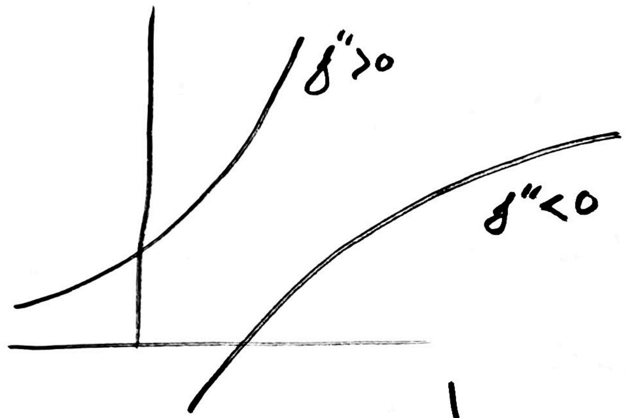
Leibniz notasjon for derivasjon.

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \frac{df}{dx} = \frac{d}{dx} f$$

$$f''(x) = \frac{d^2}{dx^2} f = \frac{d^2 f}{dx^2}$$

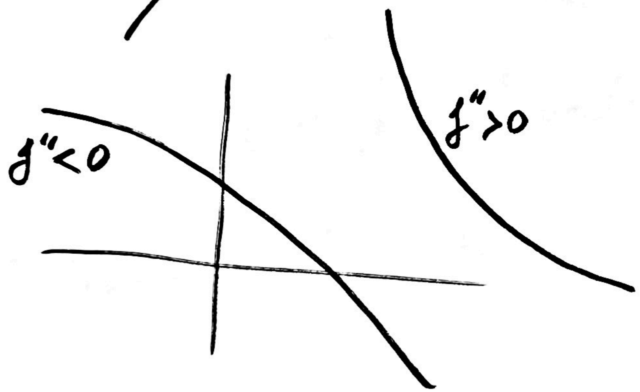
$$f^{(n)}(x) = \frac{d^n}{dx^n} f = \frac{d^n f}{dx^n}.$$

$f' > 0$



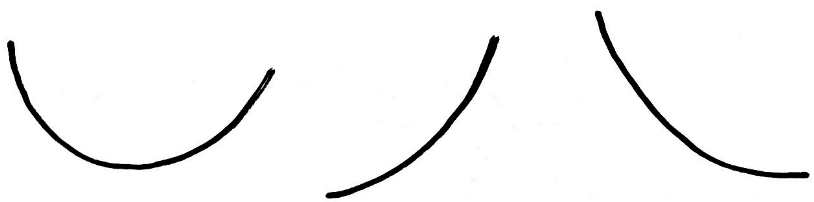
f vokser

$f' < 0$



f avtar.

$f'' > 0$ f' vokser



konkav opp
(konveks)

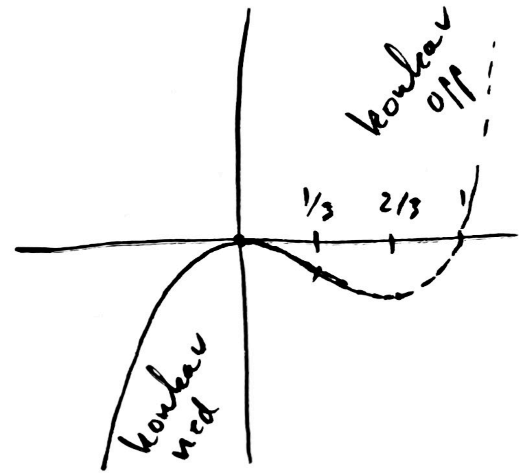
$f'' < 0$ f' avtar

konkav ned
(konkav)

Eks $f(x) = x^3 - x^2$ hvor er $f(x)$ konkav opp og konkav ned?

$f'(x) = 3x^2 - 2x = x(3x-2)$

$f''(x) = 6x - 2 = 2(3x-1) = 6(x - \frac{1}{3})$



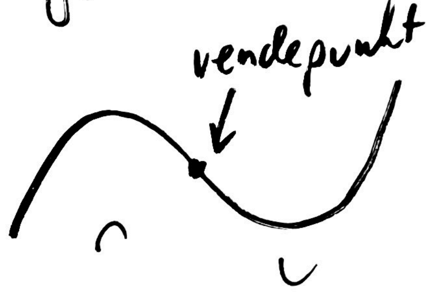
$f''(x) > 0$ for $x > \frac{1}{3}$ konkav opp

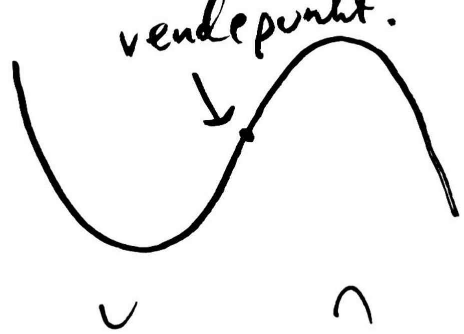
$f''(x) < 0$ for $x < \frac{1}{3}$ konkav ned.

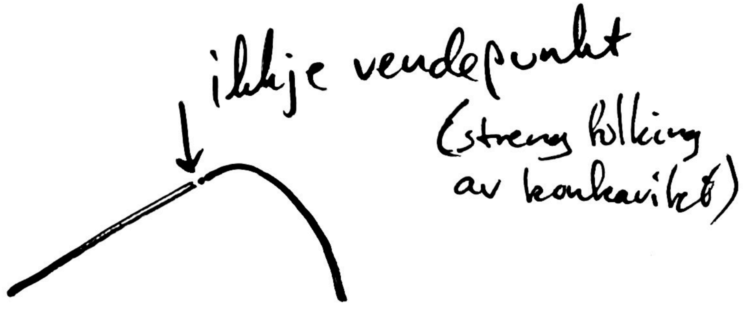
($f'(x) = 0$ for $x = 0$ og $x = \frac{2}{3}$)

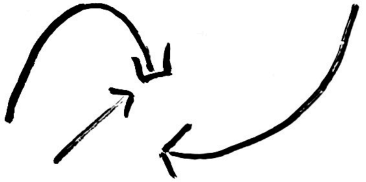
$(x, f(x))$ på grafen til $f(x)$ er et <u>vendepunkt</u>

hvis $f(x)$ skifter konkavitet og f er kont. i x

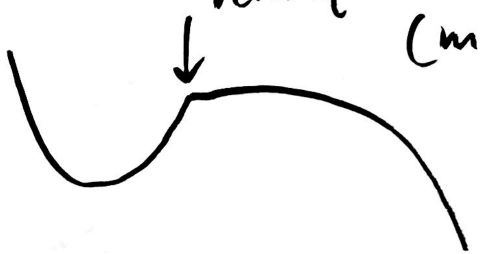




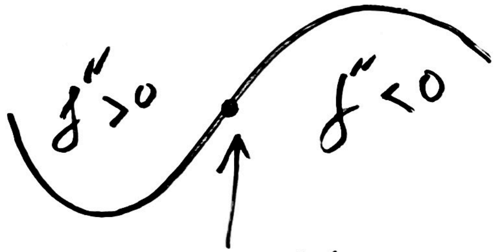




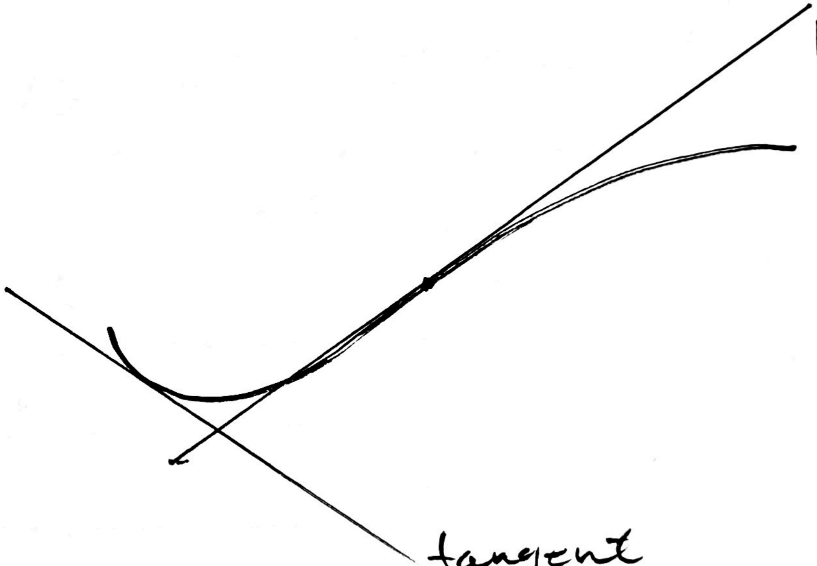
ikkje vendepunkt (diskontinuerlig)



vendepunkt (men ikkje deriverbar)



I vendepunkter er $f''(x)=0$
hvis f er dobbelt deriverbar.



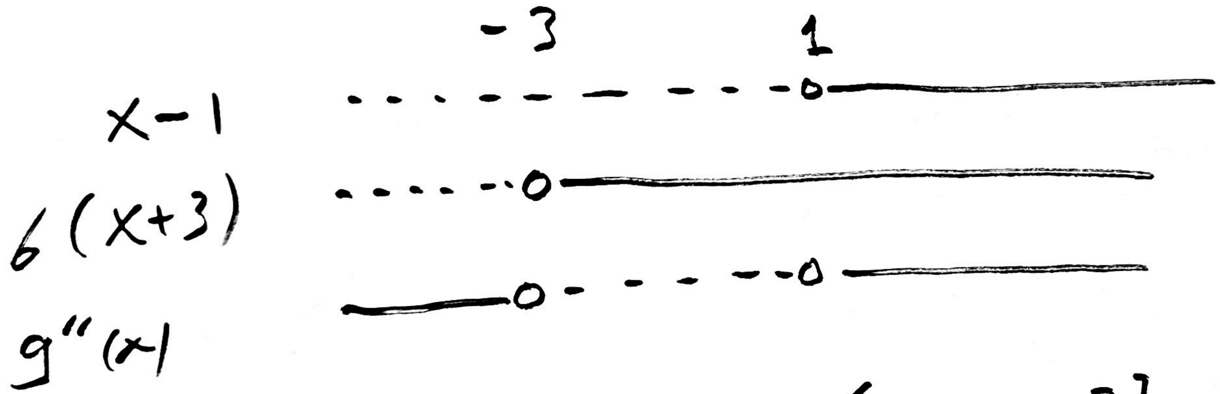
opg Bestem konkaviteten til

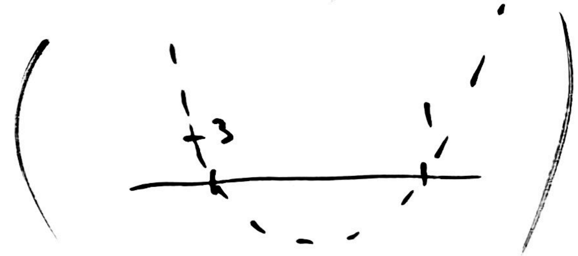
$$g(x) = \frac{x^4}{2} + 2x^3 - 9x^2 + 3x - 7.$$

$$g'(x) = 2x^3 + 6x^2 - 18x + 3$$

$$g''(x) = 6x^2 + 12x - 18 = 6(x^2 + 2x - 3)$$

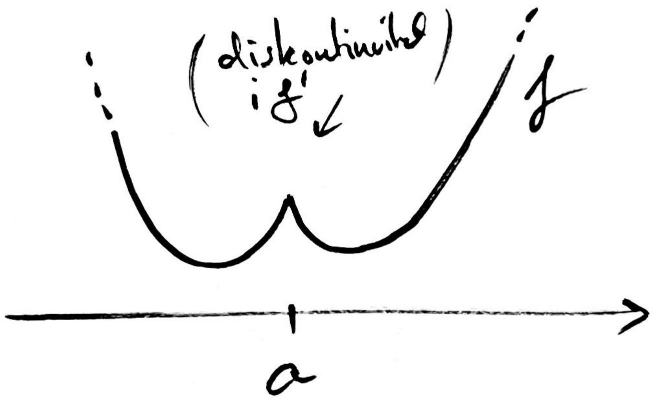
$$= 6(x+3)(x-1)$$

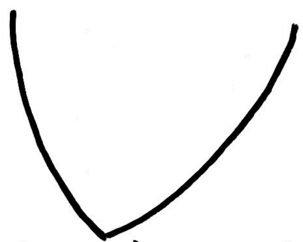




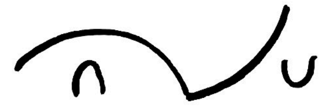
$g(x)$ er konkav opp i $\langle -\infty, -3]$ og i $[1, \infty\rangle$

konkav ned i $[-3, 1]$.





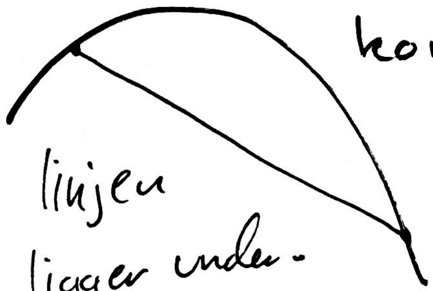
konkav opp.



f', f'' ikke definert i vendepunktet?

ikke konkav i $\mathbb{R}$ men konkav i $\langle -\infty, a]$ og i $[a, \infty\rangle$.

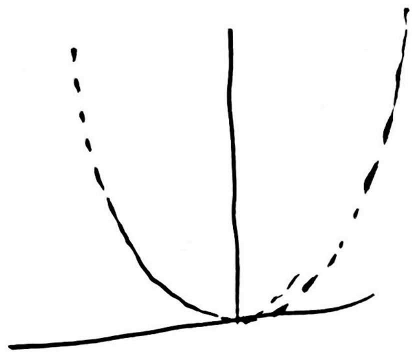
konkav opp



konkav ned.

$f = x^2$

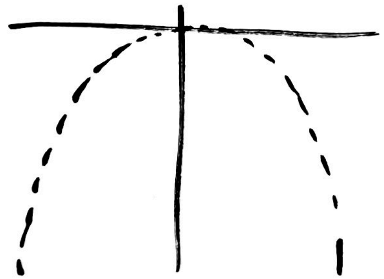
$f''(x) = 2$



konkav opp

$f = -x^2$

$f'' = -2$

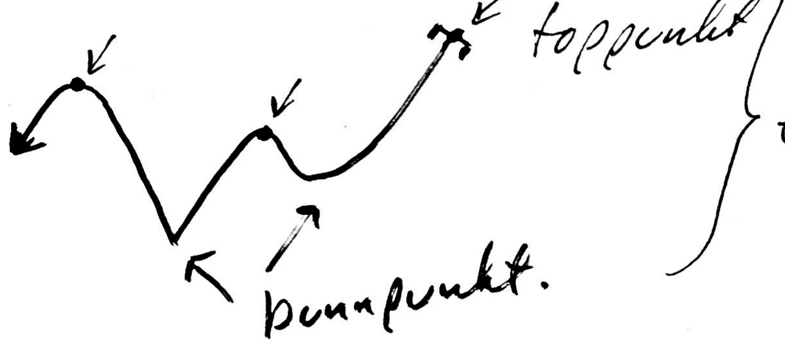


konkav ned.

$\frac{1}{R}$ krumning.

sirkelen
som følger
grafen best mulig
nær et punkt på grafen.

7.1 Funksjonsdrøfting, ekstremalpunkt.

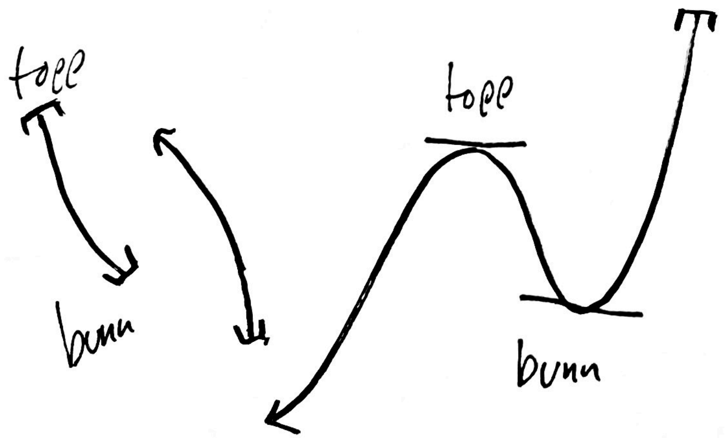


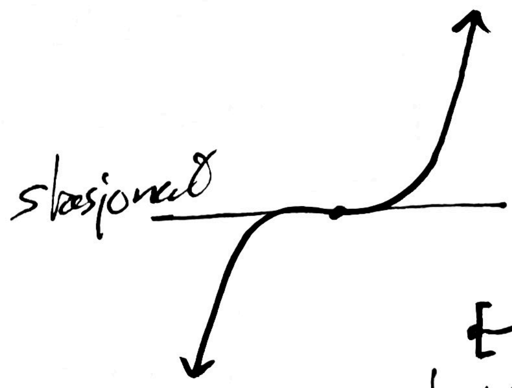
Kritiske punkt : $x \in D_f$ slik at 1) $f'(x) = 0$ (stasjonære punkt)

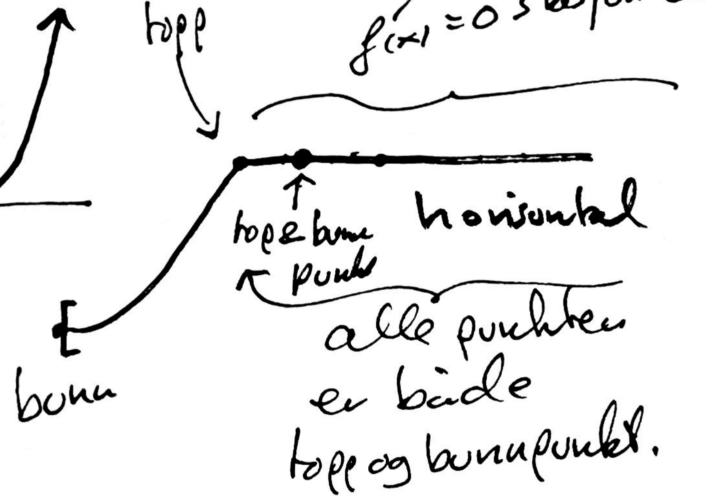
2) $f'(x)$ eksisterer ikkje

3) endepunkt.

Ekstremalpunkt finner vi blant de kritiske punktene.





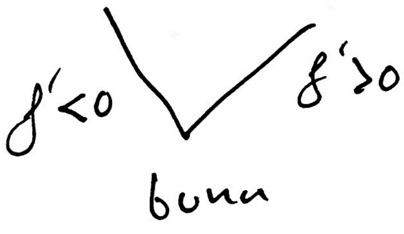


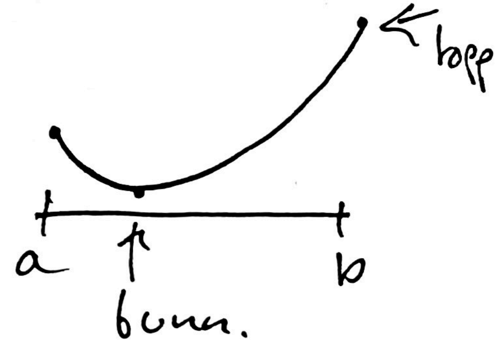
$f'(a) = 0$ og $f'(x) < 0$ for $x < a$, $f'(x) > 0$ — $x > a$ } $(a, f(a))$ bunnpunkt

(tilstrekkelig at f er kont i a)

$f'(a) = 0$ og $f'(x) > 0$ for $x < a$, $f'(x) < 0$ for $x > a$ } toppunkt

nær a.

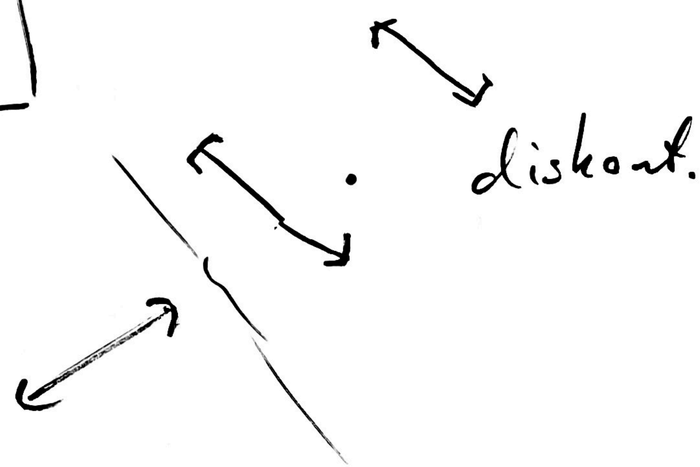




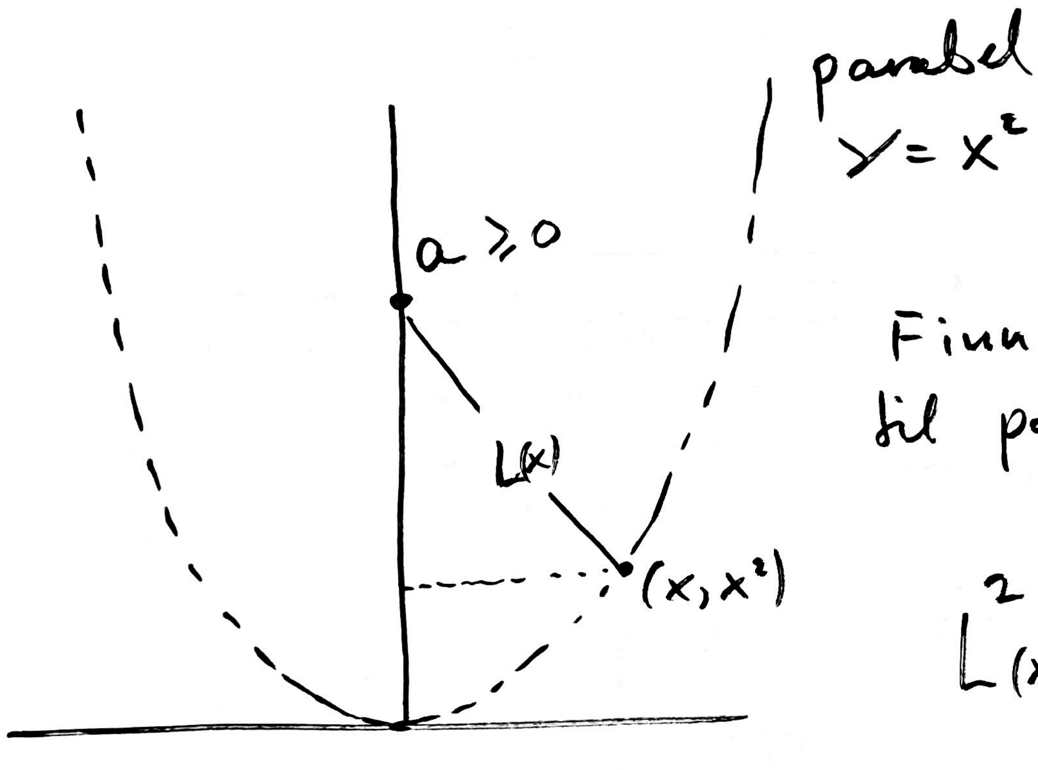
Resultat: Ekstremalverdisetningen

f kontinuerlig på $[a, b]$

Da har f både globale topp og bunnpunkt



på (a, b) $f(x) = x$ har ikke topp/bunnpunkt.



Finn korteste avstand til parabelen $(x \geq 0)$

$$L^2(x) = (a - x^2)^2 + x^2$$
$$= x^4 - 2a x^2 + x^2.$$

$L(x)$ (≥ 0) er minst når $L^2(x)$ er minst.

$$\left(L^2(x)\right)' = 4x^3 + 2x(1-2a).$$
$$= 2x\left(2x^2 + (1-2a)\right)$$

Kritiske punkt: $\left(L^2(x)\right)' = 0$: $x = 0$

$$x^2 + \frac{1-2a}{2} = 0$$

$x^2 = a - \frac{1}{2}$ så $x = \pm\sqrt{a - \frac{1}{2}}$ $a \geq 1/2$.

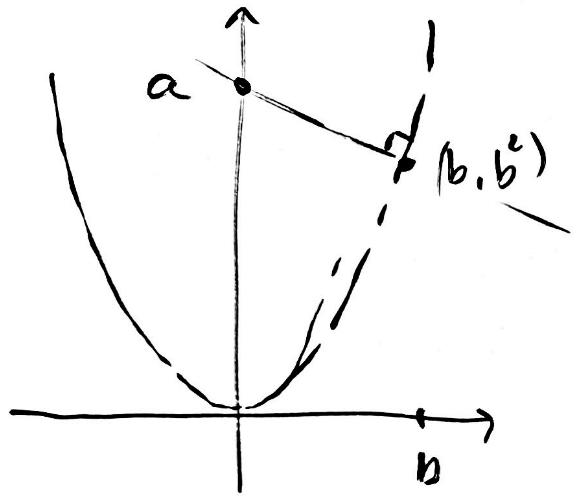
$L(0) = a$

$L\left(\sqrt{a-\frac{1}{2}}\right) = \sqrt{\left(a - \left(\sqrt{a-\frac{1}{2}}\right)^2\right)^2 + \sqrt{a-\frac{1}{2}}^{\,2}}$ $a \geq \frac{1}{2}$

$= \sqrt{\left(\frac{1}{2}\right)^2 + a - \frac{1}{2}} = \underline{\sqrt{a - \frac{1}{4}}}$

Minste avstand $\begin{cases} a & 0 \leq a \leq \frac{1}{2} \\ \sqrt{a-\frac{1}{4}} & \frac{1}{2} < a \end{cases}$ skjer i $x = 0$

— $\pm\sqrt{a-\frac{1}{2}}$



Normallinjen

$$y = -\frac{1}{2b}(x-b) + b^2$$

$$y = -\frac{x}{2b} + \frac{1}{2} + b^2$$

treffer y-aksen i $b^2 + \frac{1}{2}$.

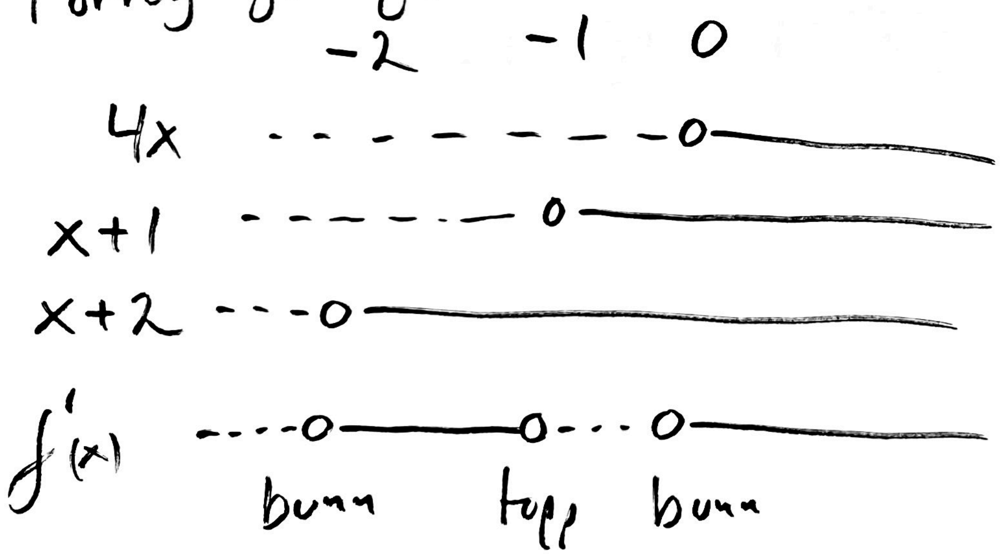
$a = b^2 + \frac{1}{2}$ $\qquad$ $a \geq \frac{1}{2}$ og $b = \pm\sqrt{a - \frac{1}{2}}$.

Eks $f(x) = x^4 + 4x^3 + 4x^2 - 1$

$$f'(x) = 4x^3 + 4\cdot 3x^2 + 4\cdot 2x$$
$$= 4x(x^2 + 3x + 2)$$
$$= 4x(x+2)(x+1)$$

$f'(x) = 0$ for $\underbrace{x = 0, -1, -2.}_{\text{Kritiske punkt.}}$

Fortegn for $f'(x)$:



Toppunkt:

$(-1, 0)$

Bunnpunkt:

$(0, -1)$

$(-2, -1)$

$$f''(x) = \left(4(x^3+3x^2+2x)\right)'$$
$$= 4(3x^2+6x+2)$$

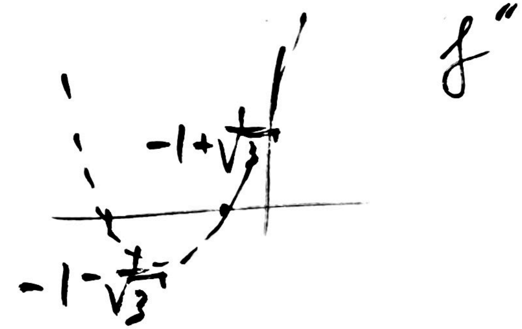
nullpunkt til f''

$$x = \frac{-6 \pm \sqrt{36-6\cdot 4}}{2\cdot 3}$$

$$x = \frac{-6 \pm 2\sqrt{3}}{2\cdot 3}$$

$$x = \frac{-3 \pm \sqrt{3}}{3}$$

$$x = \underline{-1 \pm \frac{1}{\sqrt{3}}}$$



f er konkav opp $\left\langle -\infty, -1-\frac{1}{\sqrt{3}}\right]$

og i $\left[-1+\frac{1}{\sqrt{3}}, \infty\right\rangle$

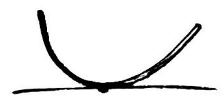
f konkav ned $\left[-1-\frac{1}{\sqrt{3}}, -1+\frac{1}{\sqrt{3}}\right]$

Vendepunkt

$$\left(-1 \pm \frac{1}{\sqrt{3}}, f\left(-1 \pm \frac{1}{\sqrt{3}}\right)\right).$$

Dobbelderivert testen.

$f'(a) = 0$ og $f''(a) > 0$: bunnpunkt



$f'(a) = 0$ og $f''(a) < 0$: toppunkt

$f''(a) = 0$: ingen konklusjon fra testen.

Foregående eks. Kritiske punkt $0, -1, -2$. $(f'(x) = 0)$

$f''(-1) = 4(3(-1)^2 + 6(-1) + 2) = -4 < 0 \Rightarrow$ toppunkt

$f''(0) = 8 > 0 \Rightarrow$ bunnpunkt

$f''(-2) = 4(3(-2)^2 + 6(-2) + 2) = 8 > 0 \Rightarrow$ bunnpunkt.