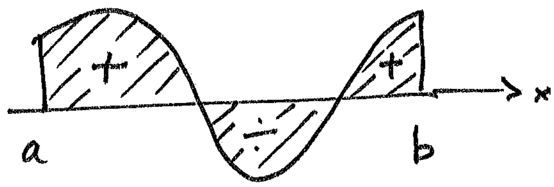
19.04.2012 16.3 Bestemte integral

(1) $\int_a^b f(x)\,dx$



Det bestemte integralet av $f(x)$ fra a til b

Hvis $f(x)$ er kontinuerlig på $[a,b]$, da finnes det en antiderivert $F(x)$ til $f(x)$ og

$$\int_a^b f(x)\,dx = F(b) - F(a) \quad \text{(fundamental teoremet)}.$$

Substitusjon og bestemte integral. $f(x) = F'(x)$

kjerneregelen: $(F(u(x)))' = f(u(x)) \cdot u'(x)$

$$\int f(u(x)) \cdot u'dx = F(u(x)) + c = \int f(u)\,du.$$

$u'dx$ erstattes av du

$$\int_a^b f(u(x))\,u'(x)\,dx = F(u(x))\Big|_a^b = F(u(b)) - F(u(a))$$

$$= \int_{u(a)}^{u(b)} f(u)\,du$$

En vanlig feil er å bruke $\cancel{\int_a^b f(u)\,du}$.

② $\int_2^3 \overset{u'}{2x}\overset{u^3}{(x^2-4)^3}\,dx$

$u = x^2-4$

$u' = 2x$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

$= \int_{u(2)}^{u(3)} u^3\,du$

$= \int_0^5 u^3\,du$

$= \frac{u^4}{4}\Big|_0^5 = \frac{5^4}{4} - \frac{0}{4}$

$= \underline{\underline{\frac{625}{4}}}$

Alternativt: $\int \overset{u'}{2x}\overset{u^3}{(x^2-4)^3}\,dx$

$= \int u' u^3\,dx = \int u^3\,du$

$= \frac{u^4}{4} + c = \frac{(x^2-4)^4}{4} + c$

Så $\frac{(x^2-4)^4}{4}$ er en antiderivert til $2x(x^2-4)^3$.

$\int_2^3 2x(x^2-4)^3\,dx = \frac{(x^2-4)^4}{4}\Big|_2^3 = \frac{5^4}{4} - \frac{0}{4}$

$= \underline{\underline{\frac{625}{4}}}$

oppg 3

$$\int_1^2 \frac{1}{3x+1}\,dx$$

$U = 3x+1$

$U' = 3$

$$\int_1^2 \frac{1}{U}\cdot\frac{U'}{3}\,dx \qquad \left(\frac{U'}{3}=1\right)$$

$U(1) = 4$

$U(2) = 7$

$$= \frac{1}{3}\int_4^7 \frac{1}{U}\,dU$$

$\frac{dU}{dx} = 3$

$dU = 3dx$

$\frac{1}{3}dU = dx$

$$= \frac{1}{3}\ln U \Big|_4^7$$

$$= \frac{1}{3}(\ln 7 - \ln 4)$$

$$= \underline{\underline{\frac{1}{3}\ln\left(\frac{7}{4}\right)}}$$

oppg

$$\int_0^{\pi} \overset{V}{t}\,\overbrace{\sin(2t)}^{u'}\,dt$$

(eksamen 2011)

$V' = 1$

$U = \frac{-\cos(2t)}{2}$

$$= t\left(-\frac{\cos(2t)}{2}\right)\Big|_0^{\pi}$$

$$- \int_0^{\pi} 1\left(\frac{-\cos(2t)}{2}\right)dt$$

$$= \pi\left(\frac{-1}{2}\right) + \frac{1}{2}\int_0^{\pi}\cos(2t)\,dt$$

$$= \frac{-\pi}{2} + \frac{1}{2}\left[\frac{\sin(2t)}{2}\right]_0^{\pi} = \underline{\underline{\frac{-\pi}{2}}}$$

Delvis integrasjon og bestemte integral.

(4) $(u \cdot v)' = u' \cdot v + u \cdot v'$ produktregelen for derivasjon.

$$\int u' \cdot v + u \cdot v' \, dx = u \cdot v + c$$

$$\int_a^b u' \cdot v \, dx + \int_a^b u \cdot v' \, dx = u \cdot v \Big|_a^b$$

$$= u(b) \cdot v(b) - u(a) \cdot v(a)$$

$$\int_a^b u' \cdot v \, dx = u \cdot v \Big|_a^b - \int_a^b u \cdot v' \, dx$$

Eksempel: $\int_0^1 x e^{-x} \, dx$

La $v = x$

$u' = e^{-x}$

$v' = 1$

Velger $u = -e^{-x}$

$$= -x e^{-x} \Big|_0^1 - \int_0^1 1 \cdot (-e^{-x}) \, dx$$

$$= \left(\frac{-1}{e} - 0 \right) - e^{-x} \Big|_0^1$$

$$= \frac{-1}{e} - \left(\frac{1}{e} - 1 \right) = \underline{1 - \frac{2}{e}}$$

~~Hvor kom me 2-tallet fra?~~ $\cancel{\frac{-1}{e} - \frac{1}{e} = -\left(\frac{1}{e} + \frac{1}{e}\right) = \frac{-2}{e}}$

Alternativt: $\int x e^{-x} \, dx = -x e^{-x} - e^{-x} + c$

Så $\int_0^1 x e^{-x} \, dx = -x e^{-x} - e^{-x} \Big|_0^1 = -e^{-1} - e^{-1} - (-e^0)$

$$= \underline{1 - \frac{2}{e}}$$

Eksamen 2009

(5) Finn arealet av området begrenset av grafen til $y = \frac{x^3}{\sqrt{x^2+1}}$, x-aksen og linjen $x=0$ og $x=1$.

($y \geq 0$ for $x \in [0,1]$)

Arealet $A = \int_0^1 y(x)\,dx = \int_0^1 \frac{x^3}{\sqrt{x^2+1}}\,dx$

Prøver med variabelskifte $u = x^2+1$, $x^2 = U-1$

$u' = 2x$

$$A = \int_0^1 2x \cdot \frac{x^2}{2} \cdot \frac{1}{\sqrt{U}}\,dx$$

$2x \cdot \frac{x^2}{2} = x^3$

$u' \cdot \frac{U-1}{2} = x^3$

$$= \int_0^1 \underbrace{2x}_{u'} \cdot \frac{U-1}{2} \cdot \frac{1}{\sqrt{U}}\,dx$$

$U(0) = 1$

$U(1) = 2$

$$= \int_{U(0)}^{U(1)} \frac{U-1}{2} \cdot \frac{1}{\sqrt{U}}\,dU$$

$\left(\frac{U-1}{\sqrt{U}} = \frac{U}{\sqrt{U}} - \frac{1}{\sqrt{U}}\right)$

$$= \frac{1}{2}\int_1^2 \sqrt{U} - \frac{1}{\sqrt{U}}\,dU$$

$$= \frac{1}{2}\int_1^2 U^{1/2} - U^{-1/2}\,dU = \frac{1}{2}\left[\frac{U^{3/2}}{3/2} - \frac{U^{1/2}}{1/2}\right]_1^2$$

$$= \left[\frac{U \cdot \sqrt{U}}{3} - \sqrt{U}\right]_1^2 = \sqrt{U}\left(\frac{U}{3} - 1\right)\Big|_1^2$$

$$= \sqrt{2}\underbrace{\left(\frac{2}{3} - 1\right)}_{-1/3} - \sqrt{1}\underbrace{\left(\frac{1}{3} - 1\right)}_{-2/3}$$

$$= \underline{\underline{\frac{2-\sqrt{2}}{3}}}$$

(ex. 16.14 i boka)

⑥ $\int_0^1 \frac{1}{\sqrt{1+\sqrt{t}}}\,dt$

$$\int_0^1 \frac{1}{\sqrt{u}} \frac{u'}{u'}\,dt$$

$$= \int_{u(0)}^{u(1)} \frac{1}{\sqrt{u}} \frac{1}{u'}\,du$$

$$= \int_{u(0)}^{u(1)} \frac{1}{\sqrt{u}}\, 2(u-1)\,du$$

$$= 2\int_1^2 \frac{1}{\sqrt{u}} (u-1)\,du$$

$= 4 \cdot$ (integralet i forrige eksempel)

$$= \underline{\underline{4\,\frac{(2-\sqrt{2})}{3}}}$$

$u = 1+\sqrt{t}$

$u' = \frac{1}{2\sqrt{t}}$

u' kan uttrykkes som en funksjon av u.

$\sqrt{t} = u-1$

$u' = \frac{1}{2\sqrt{t}} = \frac{1}{2(u-1)}$.

$\frac{1}{u'} = 2(u-1)$

$$\int (1+x^3)^{20}\, dx$$

spørsmål fra studentene.

$$\int (1+x^3)^2\, dx$$

$$= \int 1 + 2x^3 + x^6\, dx$$

$$= x + 2\frac{x^4}{4} + \frac{x^7}{7} + c$$

$n=2$

$$(1+x^3)^{20} = \sum_{i=0}^{20} \binom{n}{i} 1^{n-i} (x^3)^i$$

$$= \sum_{i=0}^{20} \binom{n}{i} x^{3i}$$

$$\binom{n}{i} = \frac{n!}{i!(n-i)!} \qquad n! = 1\cdot 2\cdot 3 \cdots n$$

$$\int (1+x^3)^{20}\, dx = \sum_{i=0}^{20} \binom{n}{i} \frac{x^{3i+1}}{3i+1} + c$$

$$= x + 5x^4 + \frac{190}{7}x^7 + 114x^{10} + \frac{4845}{13}x^{13} + 969x^{16}$$
$$+ 2040x^{19} + \frac{77520}{22}x^{22} + \frac{125970}{25}x^{25} + \frac{167960}{28}x^{28}$$
$$+ \frac{184756}{31}x^{31} + 4940x^{34} + \frac{125970}{37}x^{37} + 1938x^{40}$$
$$+ \frac{38760}{43}x^{43} + \frac{15504}{46}x^{46} + \frac{4845}{49}x^{49} + \frac{1140}{52}x^{52}$$
$$+ \frac{190}{55}x^{55} + \frac{10}{29}x^{58} + \frac{1}{61}x^{61} + c$$

$\int \sin 2t \, dt$

$w = 2t$

$dw = 2dt$

$\frac{1}{2} dw = dt$

$= \int \sin(w) \frac{1}{2} dw$

$= \frac{1}{2} \int \sin(w) dw$

$= \frac{-1}{2} \cos(w) + c$

$= \underline{\frac{-1}{2} \cos(2t) + c}$

Kladd