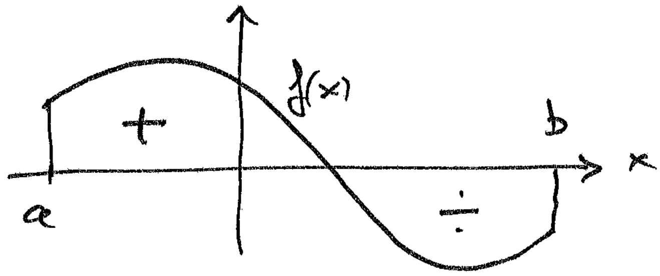
10 april 2012 Fundamentalteoremet i kalkulus

a, b reelle tall.

1 Det bestemte integralet av $f(x)$ fra a til b (med hensyn til x)

$$\int_a^b f(x)\,dx$$

er



arealet (med fortegn) mellom grafen til $f(x)$ og x-aksen fra a til b.

(begrensa)

Bestemte integral eksisterer for en stor klasse funksjoner.

$$\int_a^x f(t)\,dt \quad \text{er en funksjon i } x.$$

Fundamentalteoremet i kalkulus.

La $f(x)$ være en <u>kontinuerlig</u> funksjon på $[a, b]$ da er

$$\boxed{\frac{d}{dx}\int_a^x f(t)\,dt = f(x) \qquad x \in [a,b]}$$

Så $\int_a^x f(t)\,dt$ er en antiderivert til $f(x)$.

(2)

La $F(x)$ være en antiderivert til $f(x)$

(antideriverte er bestemt opp til å legge til en konstant)

$$F(x) + c = \int_a^x f(t)\,dt$$ for en konstant c.

Finner c: setter $x = a$

da er $\int_a^a f(t)\,dt = 0$

så $F(a) + c = 0$

og $c = -F(a)$.

$$\int_a^x f(t)\,dt = F(x) - F(a)$$

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setter $x = b$

$$\int_a^b f(t)\,dt = F(b) - F(a)$$

$$= F(x)\Big|_a^b \quad \text{(notasjon)}$$

Dette gir en nyttig metode til å evaluere bestemte integral (av kontinuerlige funksjoner).

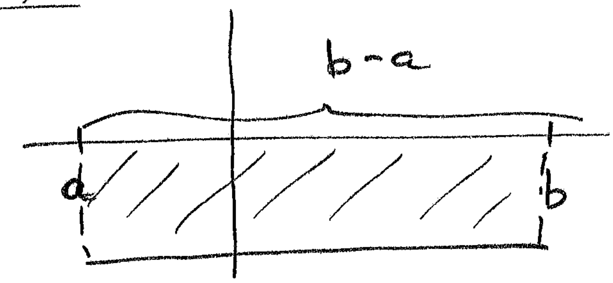
eksempler

3

$$\int_a^b k\,dx = \underbrace{k\cdot x + c}_{\text{antiderivert til } k} \Big|_a^b$$

$$= (k\cdot b + c) - (k\cdot a + c)$$

$$= k(b-a)$$

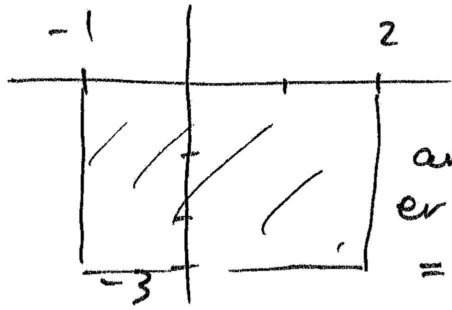


$k < 0$ arealet er $(-k)(b-a)$

$-$ arealet $k(b-a)$

$$\int_{-1}^{2} -3\,dx$$

$$= -9$$

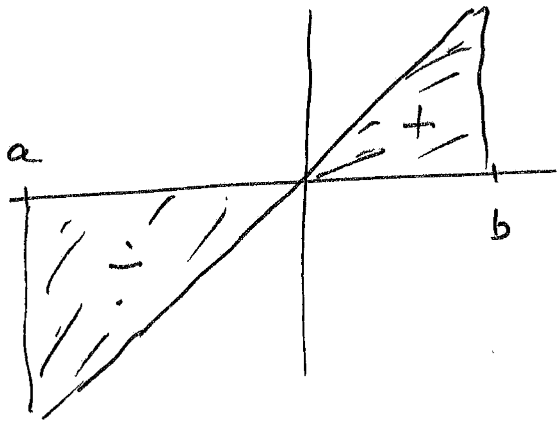


arealet er $3\cdot 3$ $= 9$

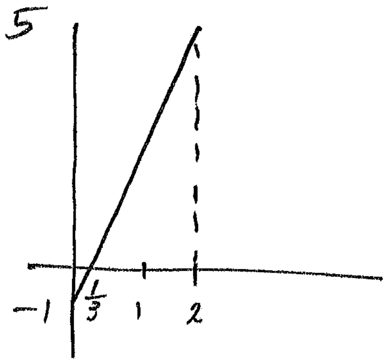
$$\int_a^b x\,dx$$

$$= \frac{x^2}{2}\Big|_a^b$$

$$= \frac{b^2}{2} - \frac{a^2}{2}$$



oppg. $\int_0^2 3x - 1 \, dx$



4

$$= \frac{3x^2}{2} - x \Big|_0^2$$

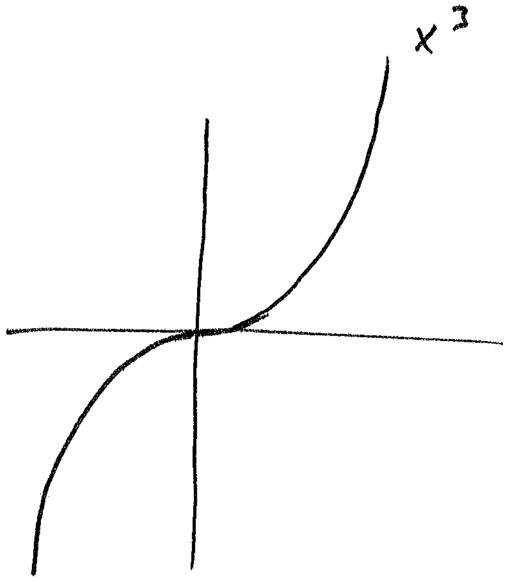
$$= \frac{3(2^2)}{2} - 2 - 0$$

$$= 6 - 2 = \underline{\underline{4}}$$

oppg. Bestem 1) $\int_a^b x^3 \, dx$.

2) $\int_{-a}^{a} x^3 \, dx$

3) $\int_0^b x^3 \, dx$



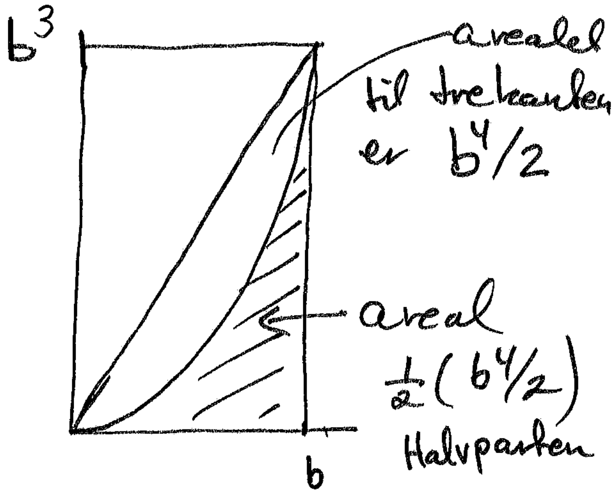
1) $\int_a^b x^3 \, dx = \frac{x^4}{4} \Big|_a^b = \frac{b^4}{4} - \frac{a^4}{4}$

$$= \underline{\frac{1}{4}(b^4 - a^4)}$$

2) $\int_{-a}^{a} x^3 \, dx = \frac{a^4}{4} - \frac{(-a)^4}{4} = \frac{a^4}{4} - \frac{a^4}{4} = 0$

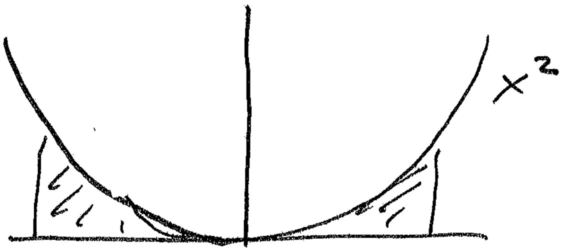
$= 0$ (siden x^3 er en odde funksjon)

3) $\int_0^b x^3 \, dx = \frac{b^4}{4}$



eks.

5 $\displaystyle\int_a^b x^2\,dx$

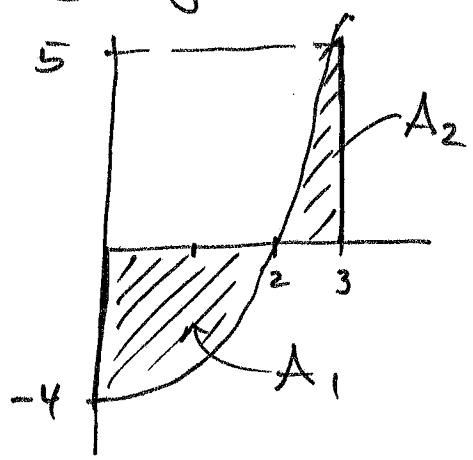


(x^2 er en jevn funksjon
siden $(-x)^2 = (-1)^2 x^2 = x^2$)

$$\int_a^b x^2\,dx = \frac{x^3}{3}\Big|_a^b = \frac{b^3}{3} - \frac{a^3}{3}.$$

$$\int_{-2}^{2} x^2\,dx = \frac{2^3}{3} - \frac{(-2)^3}{3}$$
$$= \frac{2^3}{3} - \frac{-2^3}{3}$$
$$= \frac{8}{3} + \frac{8}{3} = \underline{\frac{16}{3}}.$$

Eksempel Finn <u>arealet</u> avgrenset av
$f(x) = x^2 - 4$, x-aksen og linjene
$x = 0$ og $x = 3$



$$A_1 = -\int_0^2 f(x)\,dx$$

$$A_2 = +\int_2^3 f(x)\,dx$$

$F(x) = \frac{x^3}{3} - 4x$ er en antiderivert til $f(x)$.

$F(0) = 0$, $F(2) = \frac{2^3}{3} - 4\cdot 2 = \frac{8}{3} - 8 = 8(\frac{1}{3} - 1) = \frac{-16}{3}$

$F(3) = \frac{3^3}{3} - 4\cdot 3 = 9 - 12 = -3$

$$A_1 = -\int_0^2 f(x)\,dx = -\left(F(2) - F(0)\right)$$

(6)

$$= -F(2) = \frac{16}{3} = \underline{5 + \tfrac{1}{3}}$$

$$A_2 = \int_2^3 f(x)\,dx = F(3) - F(2)$$

$$= -3 - \left(\frac{-16}{3}\right) = \frac{16}{3} - 3$$

$$= \frac{7}{3} = \underline{2 + 1/3}$$

Arealet er $A_1 + A_2$

$$= 5 + \tfrac{1}{3} + 2 + \tfrac{1}{3} = \underline{\underline{7 + \tfrac{2}{3}}}$$

(Det bestemte integralet $\int_0^3 f(x)\,dx$

$$= -A_1 + A_2 = -\left(5 + \tfrac{1}{3}\right) + 2 + 1/3 = \underline{\underline{-3}}$$

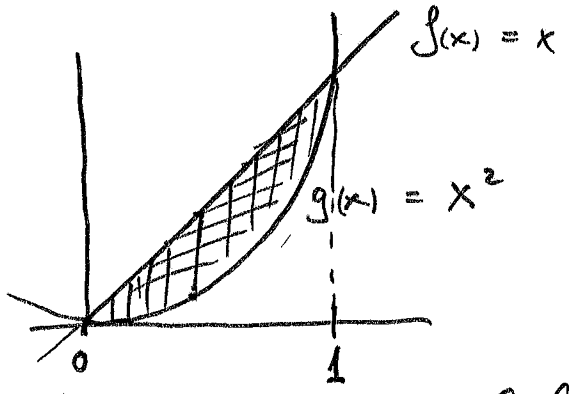
$$= -5 - \tfrac{1}{3} + 2 + \tfrac{1}{3}$$)

(Som forventet siden

$$\int_0^3 f(x)\,dx = F(3) - F(0)$$

$$= -3 - 0 = -3.$$)

(7)



$$\left(\begin{array}{c} x - x^2 = x(1-x) \\ \geq 0 \\ \text{for} \quad x \in [0,1] \end{array} \right)$$

Hva er arealet til regionen avgrenset av $f(x) = x$ og $g(x) = x^2$?

$$A = \int_0^1 f(x) - g(x)\, dx$$

$$= \int_0^1 f(x)\, dx - \int_0^1 g(x)\, dx$$

$$= \int_0^1 (x - x^2)\, dx$$

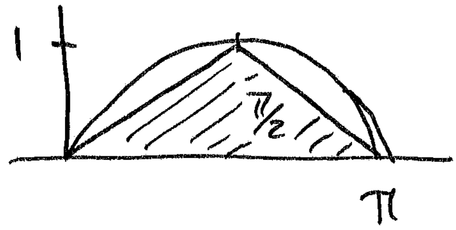
$$= \frac{x^2}{2} - \frac{x^3}{3} \Big|_0^1 = \frac{1}{2} - \frac{1}{3} - 0$$

$$= \underline{\underline{\frac{1}{6}}}$$

Arealet til regionen avgrenset av $f(x) = x$ og $g(x) = x^2$ er $\underline{\underline{\frac{1}{6}}}$

$$\int_0^{\pi} \sin x \, dx$$

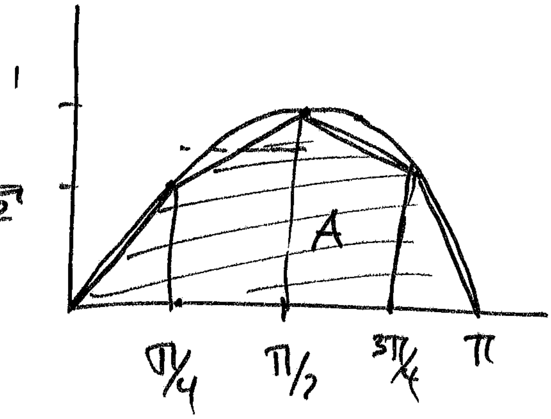
8



$$\frac{\pi \cdot 1}{2} < \int_0^{\pi} \sin x \, dx$$

≈ 1.57

$= \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$



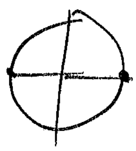
Arealet til tilnærmingen : A.

$$2\left[\frac{\pi}{4}\left(\frac{1}{\sqrt{2}} \cdot \frac{1}{2}\right) + \frac{\pi}{4}\left(\frac{\frac{1}{\sqrt{2}}+1}{2}\right)\right]$$

$$= \frac{\pi}{2}\left(\frac{1}{2} \cdot \frac{1}{\sqrt{2}} + \frac{1}{2}\left(\frac{1}{\sqrt{2}} + 1\right)\right]$$

$$= \frac{\pi}{4}\left[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + 1\right] = \frac{\pi}{4}\left[\frac{2}{\sqrt{2}} + 1\right] = \frac{\pi}{4}\left[\sqrt{2} + 1\right]$$

$\approx 1.89..$



$$\int_0^{\pi} \sin x \, dx = -\cos x \Big|_0^{\pi}$$

$$= -\underbrace{\cos(\pi)}_{-1} - (-\underbrace{\cos(0)}_{1})$$

$$= 1 + 1$$

$$= \underline{2}$$

eksempel $\int_0^x \sqrt{t}\,dt$

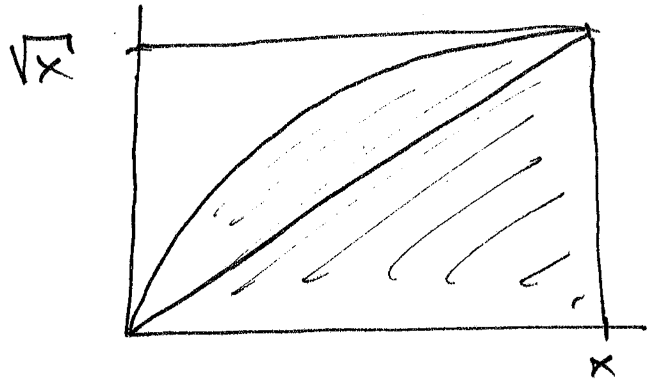
(9)

$$= \int_0^x t^{1/2}\,dt$$

$$= \frac{t^{3/2}}{3/2}\Big|_0^x$$

$$= \frac{x^{3/2}}{3/2} - 0$$

$$= \frac{2}{3}x^{3/2} = \underline{\frac{2}{3}x\cdot\sqrt{x}}.$$



$$\frac{1}{2}x\sqrt{x} < \int_0^x \sqrt{t}\,dt < x\sqrt{x}$$

(areal trekant) (areal rektangel)

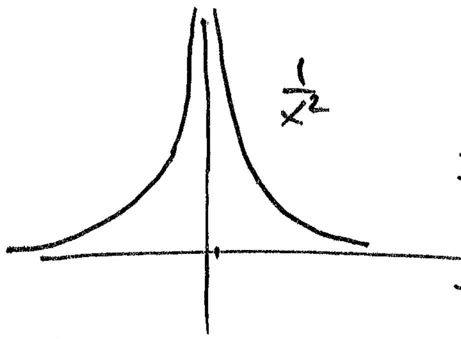
høyde $1/d^2$, bredde d, med areal $\frac{1}{d}$ for alle $d>0$.

oppg. $\int_{-2}^{1} \frac{1}{x^2}\,dx$

Ukritisk bruk av fund. teoremet:

$$\int_{-2}^{1} x^{-2}\,dx = -x^{-1}\Big|_{-2}^{1}$$

$$= \frac{-1}{1} - \frac{-1}{-2} = -1 + \frac{1}{2} = -\frac{1}{2}.$$



GALT

$\int_0^1 \frac{1}{x^2}\,dx$ eksisterer ikke siden regionen inneholder rektangelet

$$\begin{cases} \frac{1}{x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$ er ikke kontinuerlig i $[-2, 1]$.