10. jan 2012

8.7 Derivasjon

$f(x)$ funksjon med definisjonsmengde D_f.

$f(x)$ er deriverbar i $x \in D_f$ hvis

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$ eksisterer.

Grensa kalles den deriverte til $f(x)$ i x.

Den deriverte til $f(x)$ er funksjonen

$$\frac{d}{dx} f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Alternativ notasjon $\frac{d}{dx} f(x) = \frac{df}{dx}(x) = \frac{d f(x)}{dx}$

$$f'(x) = (f(x))'$$

$$(D_x f(x))$$

I fysikk skrives den deriverte m.h.t tiden:

$$\dot{s}(t) = \frac{d}{dt} s(t)$$

$$\left(\ddot{s}(t) = \frac{d}{dt}\left(\frac{d}{dt} s(t)\right) \right)$$

$$\frac{d}{dx}(k) = 0$$ k konstant (funksjon)

$$\frac{d}{dx}(x) = 1 = 1 \cdot x^{1-1} = 1 \cdot x^0 = 1$$

$$\frac{d}{dx}(x^2) = 2 \cdot x \quad (= 2 \cdot x^{2-1} = 2 \cdot x)$$

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}(x^{-1}) = \frac{-1}{x^2} = -1 \cdot x^{-1-1}$$

② $\frac{d}{dx} x^3 = 3 \cdot x^{3-1} = \underline{3 \cdot x^2}$

Vi viser dette fra definisjonen av den deriverte

$$\frac{d}{dx} x^3 = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \left[x^3 + 3x^2 \cdot h + 3x \cdot h^2 + h^3 - x^3 \right]$$

$$= \lim_{h \to 0} \frac{1}{h} h \left(3x^2 + 3x \cdot h + h^2 \right) = \underline{3x^2}.$$

$$\left[\begin{aligned} (x+h)^3 &= (x+h)(x+h)(x+h) = (x+h)(x^2 + 2xh + h^2) \\ &= x(x^2 + 2xh + h^2) + h(x^2 + 2xh + h^2) \\ &= x^3 + 2x^2h + x \cdot h^2 + h \cdot x^2 + 2xh^2 + h^3 \\ &= x^3 + 3x^2h + 3xh^2 + h^3 \end{aligned} \right]$$

Resultat $$\boxed{\frac{d}{dx} x^n = n \cdot x^{n-1}}$$

gyldig for n et reelt tall.

Eks $\frac{d}{dx} x^7 = 7 \cdot x^{7-1} = \underline{7 \cdot x^6}$

$$\frac{d}{dx}\left(\frac{1}{x^7}\right) = \frac{d}{dx}\left(x^{-7}\right) = -7 \cdot x^{-7-1} = -7 \cdot x^{-8} = \frac{-7}{x^8}$$

$$\frac{d}{dt} t^3 = 3 \cdot t^{3-1} = \underline{3t^2}$$

③ $(x^{1-2})' = (x^{-1})' = -1 \cdot x^{-1-1} = -1 \cdot x^{-2} = \frac{-1}{x^2}$

$\frac{d}{dx}(x^{-1})^{-1} = \frac{d}{dx} x^{(-1)(-1)} = \frac{d}{dx}(x^1) = 1$

$\frac{d}{dx}\left(\left(\frac{1}{x^2}\right)^3\right) = \frac{d}{dx}\left((x^{-2})^3\right) = \frac{d}{dx}(x^{-6}) = -6 \cdot x^{-6-1}$
$= -6 \cdot x^{-7} = \underline{\frac{-6}{x^7}}$

$\frac{d}{dx}(1 \cdot x \cdot x^2 \cdot x^3 \cdot x^4) = \frac{d}{dx}(x^{0+1+2+3+4})$
$= \frac{d}{dx}(x^{10}) = \underline{10 \cdot x^9}$

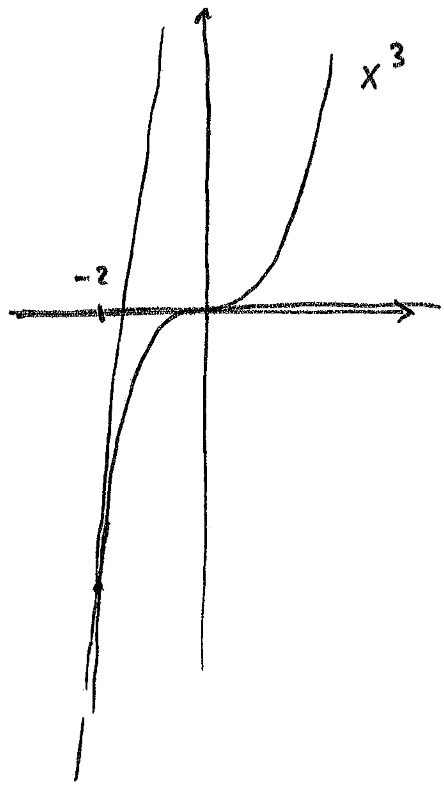
$\frac{d}{dx}\left(\frac{x^2}{x^4} \cdot x^{-1}\right) = \frac{d}{dx}\left(\frac{1}{x^2} \cdot x^{-1}\right) = \frac{d}{dx}(x^{-2} \cdot x^{-1})$

$\frac{d}{dx}(x^{-3}) = -3x^{-3-1} = -3x^{-4} = \underline{\frac{-3}{x^4}}$

---

opg. Finn likningen til tangene til $f(x) = x^3$ i 1) $(0,0)$

2) $(-2,-8)$



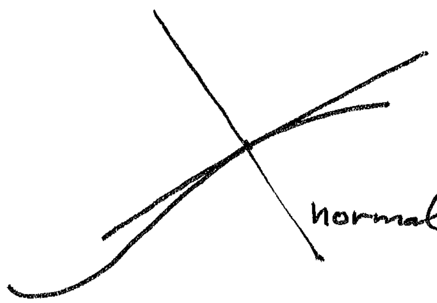
$f'(x) = 3x^2$

$f'(0) = 0$ $\qquad f'(-2) = 3(-2)^2 = 12$.

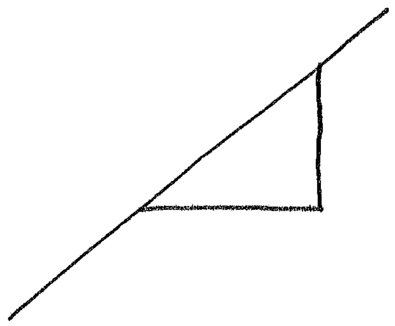
1) Tangentlinjen er x-aksen :
$y = 0$

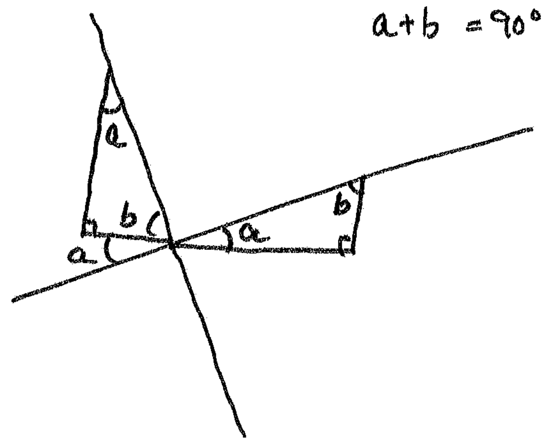
2) $y = 12(x-(-2)) + (-8)$
$= 12x + 24 - 8$
$\underline{y = 12x + 16}$

④ Normallinjen til grafen til $f(x)$ i punktet $P = (a, f(a))$ er linjen gjennom P som står vinkelrett på tangentlinja

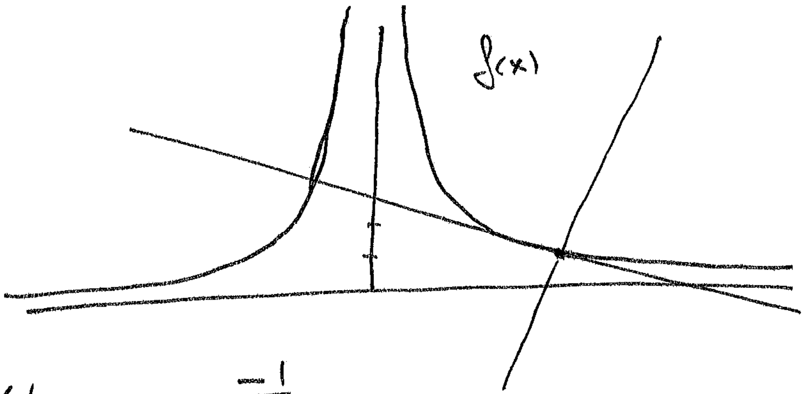


Hvis en linje har stigningstall a ($\neq 0$) så har en normal linje stigningstall $\frac{-1}{a}$.





opg. Finn tangentlinja og normal linja til $f(x) = \frac{1}{x^2}$ i punktet $(2, \frac{1}{4})$.



$$f'(x) = \left(\frac{1}{x^2}\right)' = \left(x^{-2}\right)'$$

$$= -2 \cdot x^{-3} = \frac{-2}{x^3}.$$

$$f'(2) = \frac{-2}{2^3} = \frac{-1}{4}$$

stigningstallet til tangentlinjen er $\frac{-1}{4}$

normalen er $\frac{-1}{(-1/4)} = 4$

Tangentlinjen : $y = \frac{-1}{4}(x-2) + \frac{1}{4} = \frac{-x}{4} + \frac{3}{4} = \frac{3-x}{4}$

Normallinja : $y = 4(x-2) + \frac{1}{4} = 4x - 8 + \frac{1}{4}$

9.1 Derivasjonsregler

Derivasjon er en lineær operasjon

$$(f(x) + g(x))' = (f(x))' + (g(x))'$$

$$(k \cdot f(x))' = k \cdot (f(x))' \qquad k \text{ konstant.}$$

Bevis del 1

$$\underline{(f(x)+g(x))'} = \lim_{h\to 0} \frac{(f(x+h)+g(x+h)) - (f(x)+g(x))}{h}$$

$$= \lim_{h\to 0} \left(\frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h} \right)$$

$$\overset{\text{grensesetning}}{=} \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$$

$$= \underline{f'(x) + g'(x)}$$

del 2

$$\underline{(k \cdot f(x))'} = \lim_{h\to 0} \frac{k\cdot f(x+h) - k\cdot f(x)}{h}$$

$$= \lim_{h\to 0} k \cdot \frac{f(x+h)-f(x)}{h}$$

$$\overset{\text{grensesetningene}}{=} k \cdot \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \underline{k \cdot f'(x)}$$

⑥ Vi kan nå derivere alle polynomer

eks
$$\frac{d}{dx}\left(5x^2+3x-1\right)$$
$$= \frac{d}{dx}(5x^2) + \frac{d}{dx}(3x) + \frac{d}{dx}(-1)$$
$$= 5\left(\frac{d}{dx}x^2\right) + 3\left(\frac{d}{dx}x\right) + -1\ \frac{d}{dx}(1)$$
$$= 5\cdot 2x + 3\cdot 1 + -1\cdot 0$$
$$= \underline{10x + 3}$$

—

$$\left(7x^5 + 2x^{1000} - 37x^2 + 12\right)'$$
$$= 7(x^5)' + 2\cdot(x^{1000})' - 37(x^2) + 0$$
$$= 7\cdot 5x^4 + 2\cdot 1000\cdot x^{999} - 37\cdot 2x$$
$$= \underline{35x^4 + 2000x^{999} - 74\cdot x}$$

—

$$\left(1{,}37\cdot x^3 + 2\pi\cdot x^2\right)'$$
$$= (1{,}37x^3)' + (2\pi x^2)'$$
$$= 1{,}37(x^3)' + 2\pi(x^2)'$$
$$= 1{,}37\,(3x^2) + 2\pi\cdot 2x$$
$$= (1{,}37)\cdot 3\cdot x^2 + 4\pi\cdot x$$
$$= \underline{4{,}11\cdot x^2 + 4\pi\cdot x}$$

⑦

$$\left(1{,}5x^2 + 4x^{-2} + \frac{1}{x^3}\cdot 12\right)'$$

$$= 1{,}5\,(x^2)' + 4\,(x^{-2})' + 12\,(x^{-3})'$$

$$= 1.5\,(2x) + 4(-2x^{-3}) + 12(-3x^{-4})$$

$$= 3x - 8x^{-3} - 36x^{-4}$$

$$= \underline{3x - \frac{8}{x^3} - \frac{36}{x^4}}\,.$$

$$\left(\pi + 3.14\,x^{-3}\cdot x^{-2} + \frac{x^2}{3x^2} - 17\right)'$$

$$= (\pi)' + 3{,}14\,(x^{-5})' + \left(\frac{1}{3}\right)' + (-17)'$$

$$= 0 + 3.14\,(-5\cdot x^{-6}) + 0 + 0$$

$$= 3.14\,(-5)\cdot x^{-6} = -15.7\cdot x^{-6} = \underline{\underline{\frac{-15.7}{x^6}}}$$

Finn en funksjon $F(x)$ slik at

$F'(x) = x^2$ og $F(0) = 0$.

$$(x^3)' = 3x^2$$

$$\frac{1}{3}(x^3)' = \left(\frac{1}{3}x^3\right)' = \frac{1}{3}\cdot 3x^3 = x^3$$

$$\underline{F(x) = \frac{1}{3}x^3}$$

Opg Vis at volumet til kjeglen er $\frac{\pi R^2\cdot h}{3}$

$$\underline{R(y) = \frac{R}{h}\cdot y}$$