THE PICARD GROUP OF EQUIVARIANT STABLE HOMOTOPY THEORY

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ABSTRACT. Let G be a compact Lie group. We describe the Picard group $\mathrm{Pic}(HoG\mathscr{S})$ of invertible objects in the stable homotopy category of G-spectra in terms of a suitable class of homotopy representations of G. Combining this with results of tom Dieck and Petrie, which we reprove, we deduce an exact sequence that gives an essentially algebraic description of $\mathrm{Pic}(HoG\mathscr{S})$ in terms of the Picard group of the Burnside ring of G. The deduction is based on an embedding of the Picard group of the endomorphism ring of the unit object of any stable homotopy category $\mathscr{C}$ in the Picard group of $\mathscr{C}$.

For a compact Lie group G, the isomorphism classes of invertible G-spectra form a group, $\mathrm{Pic}(HoG\mathscr{S})$, under the smash product. Here $HoG\mathscr{S}$ is the stable homotopy category of G-spectra indexed on a complete G-universe, as defined in [21]. We shall prove the following theorem.

Theorem 0.1. *There is an exact sequence*

$$0 \longrightarrow Pic(A(G)) \longrightarrow Pic(HoG\mathscr{S}) \longrightarrow C(G).$$

Here $A(G)$ is the Burnside ring of G and $C(G)$ is the additive group of continuous functions from the space of subgroups of G to the integers, where subgroups are understood to be closed. In fact, we shall see that this is implicit in results of tom Dieck and Petrie. Moreover, they and others have also studied the image of $\mathrm{Pic}(HoG\mathscr{S})$ in $C(G)$.

In §1, we give some general results on Picard groups of categories, following up [17] and [24]. In particular, we prove the following theorem, which shows that the monomorphism of Picard groups displayed in Theorem 0.1 is formal. The notion of a "stable homotopy category" is axiomatized in [17]. There are examples in algebraic topology, algebraic geometry, representation theory, and homological algebra.

Theorem 0.2. *Let $\mathscr{C}$ be a stable homotopy category, let S be the unit object, and let $R = R(\mathscr{C})$ be the ring of endomorphisms of S. There is a monomorphism of groups $c : Pic(R) \longrightarrow Pic(\mathscr{C})$. The objects in the image of c are the invertible objects that are retracts of finite coproducts of copies of S.*

In practice, stable homotopy categories are usually constructed by localizing model categories so as to invert certain objects, thus forcing them to be elements of $\mathrm{Pic}(\mathscr{C})$. For example, the equivariant stable homotopy category $HoG\mathscr{S}$ is constructed by inverting the suspension spectra of spheres S^V associated to representations of G, thus giving a homomorphism from $RO(G)$, regarded as an abelian group under addition, to $\mathrm{Pic}(HoG\mathscr{S})$; see Remark 3.6. Theorem 0.2 says that, on formal grounds, certain other objects must also be inverted. For example, the

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following parenthetical corollary is immediate from work of Morel [26] on the Morel-Voevodsky $\mathbb{A}^1$-stable homotopy category [27]; compare [24, 2.14, 4.11, 4.12].

Corollary 0.3. *Let k be a field, char $k \neq 2$, let $\mathscr{C}$ be the $\mathbb{A}^1$-stable homotopy category of k, and let $GW(k)$ be the Grothendieck-Witt ring of k. There is a monomorphism $c : Pic(GW(k)) \longrightarrow Pic(\mathscr{C})$.*

Po Hu [18] has constructed various elements in $\mathrm{Pic}(\mathscr{C})$. Her examples are genuinely exotic, in the sense that they are not in the image of $\mathrm{Pic}(GW(k))$, since calculations in motivic cohomology show that they cannot be retracts of copies of the unit object.

Returning to the equivariant stable homotopy category, in §2 we reduce the calculation of $\mathrm{Pic}(HoG\mathscr{S})$ to the study of homotopy representations of G, starting with the following slightly nonstandard definition. We shall relate this definition to previous ones in §4.

Definition 0.4. A *generalized homotopy representation* A is a finitely dominated based G-CW complex such that, for each subgroup H of G, A^H is homotopy equivalent to a sphere $S^{n(H)}$. A *stable homotopy representation* is a G-spectrum of the form $\Sigma^{-V}\Sigma^{\infty}A$, where V is a representation of G and A is a generalized homotopy representation.

Theorem 0.5. *Up to equivalence, the invertible G-spectra are the stable homotopy representations.*

In §3, we prove Theorem 0.1 by combining these algebraic and topological reductions of the problem with some arguments from the work of tom Dieck and Petrie [4, 11, 12, 13]. Theorem 0.1 gives an appropriate conceptual setting and quick new proofs for some of the main results of [12, 13].

1. The Picard group of a stable homotopy category

We assume familiarity with [24, §§1-3]. As there, let $\mathscr{C}$ be a closed symmetric monoidal category with unit object S, product $\wedge$, and internal hom functor F. Recall that the dual of an object X is $DX = F(X, S)$. Dualizable objects are discussed in [24, §2]. We are interested in invertible objects, and these are dualizable by [24, 2.9]. We have the following observation.

Lemma 1.1. *An object X is invertible if and only if the functor $(-)\wedge X : \mathscr{C} \longrightarrow \mathscr{C}$ is an equivalence of categories. If X is invertible, the canonical maps $\iota : S \longrightarrow F(X,X)$, $\eta : S \longrightarrow X \wedge DX$, and $\varepsilon : DX \wedge X \longrightarrow S$ are isomorphisms. Conversely, if ε is an isomorphism or if X is dualizable and η or ι is an isomorphism, then X is invertible.*

Proof. The first statement is clear. If X is invertible, the map

$$\mathscr{C}(-, S) \longrightarrow \mathscr{C}(-, F(X,X)) \cong \mathscr{C}(- \wedge X, X)$$

induced by ι is the isomorphism $(-) \wedge X$ given by smashing maps with X, hence ι is an isomorphism by the Yoneda lemma. When X is dualizable, the definition of η in terms of ι given in [24, 2.3] shows that ι is an isomorphism if and only if η is an isomorphism; in turn, by [24, 2.6(ii)], η is an isomorphism if and only if ε is an isomorphism. Trivially, if ε is an isomorphism, then X is invertible. □

Now assume further that the category $\mathscr{C}$ is additive. Then $R = R(\mathscr{C}) \equiv \mathscr{C}(S,S)$ is a commutative ring and $\mathscr{C}$ is enriched over the category $\mathscr{M}_R$ of R-modules, so that $\mathscr{C}(X,Y)$ is naturally an R-module. Define functors π_0, $\pi^0 : \mathscr{C} \longrightarrow \mathscr{M}_R$ by

$$\pi_0(X) = \mathscr{C}(S,X) \quad \text{and} \quad \pi^0(X) = \mathscr{C}(X,S).$$

The $\wedge$-product of maps gives a natural transformation

$$\phi : \pi_0(X) \otimes_R \pi_0(Y) \longrightarrow \pi_0(X \wedge Y).$$

The proof of Theorem 0.2 is based on application of π_0 to Künneth objects of $\mathscr{C}$, and we start with a characterization of the Künneth objects. Observe that, by [24, 2.7] and adjunction, we have canonical isomorphisms

$$\pi_0(F(X,Y) \wedge Z) \cong \pi_0(F(X, Y \wedge Z)) \cong \mathscr{C}(X, Y \wedge Z)$$

if X or Z is dualizable. In particular, if X is dualizable,

$$\pi_0(DX) \cong \pi^0(X) \quad \text{and} \quad \pi_0(DX \wedge Y) \cong \mathscr{C}(X,Y).$$

Recall that an R-module is finitely generated projective if and only if it is dualizable [24, 2.4].

Proposition 1.2. *The following conditions on a dualizable object X are equivalent, and these conditions imply that $\pi_0(X)$ is a finitely generated projective R-module. A dualizable object satisfying these conditions is said to be a* Künneth object*.*

(i) $\phi : \pi_0(DX) \otimes_R \pi_0(X) \longrightarrow \pi_0(DX \wedge X)$ *is an isomorphism.*

(ii) $\phi : \pi^0(Y) \otimes_R \pi_0(X) \cong \pi_0(DY) \otimes_R \pi_0(X) \longrightarrow \pi_0(DY \wedge X) \cong \mathscr{C}(Y,X)$ *is an isomorphism for all dualizable objects Y.*

(iii) $\phi : \pi_0(Y) \otimes_R \pi_0(X) \longrightarrow \pi_0(Y \wedge X)$ *is an isomorphism for all objects Y.*

(iv) *X is a retract of $\bigvee_{i=1}^n S$ for some integer n.*

Proof. Clearly (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) by specialization of the given isomorphisms. By [21, III.1.9], (i) $\Rightarrow$ (iii), so that (i)–(iii) are equivalent; [21, III.1.9] also shows that $\pi_0(X)$ is dualizable when these conditions hold. Write $S^{\vee n} = \bigvee_{i=1}^n S$; it is isomorphic to $\prod_{i=1}^n S$. The implication (iv) $\Rightarrow$ (iii) is clear since the conclusion of (iii) holds when $X = S^{\vee n}$ and is inherited by retracts of $S^{\vee n}$. Thus it suffices to prove that (ii) $\Rightarrow$ (iv). Assuming (ii), $\pi_0(X)$ is a finitely generated projective R-module. Therefore $\pi_0(X)$ is a direct summand and thus a retract of $R^n \cong \pi_0(S^{\vee n})$ for some n. Moreover, (ii) gives natural isomorphisms

$$\pi^0(Y) \otimes_R \pi_0(S^{\vee n}) \cong \mathscr{C}(Y, S^{\vee n}) \quad \text{and} \quad \pi^0(Y) \otimes_R \pi_0(X) \cong \mathscr{C}(Y,X).$$

Since X and $S^{\vee n}$ are dualizable, we see by the Yoneda lemma that a retraction $\pi_0(X) \longrightarrow R^n \longrightarrow \pi_0(X)$ induces a retraction $X \longrightarrow S^{\vee n} \longrightarrow X$. □

Of course, (iv) implies that X is dualizable, but the other conditions do not. The proposition shows that Künneth objects in $\mathscr{C}$ are closely related to dualizable R-modules. In particular, when $\mathscr{C} = \mathscr{M}_R$ is the category of R-modules, (iv) says that X is finitely generated projective.

Corollary 1.3. *The Künneth objects, the dualizable objects, and the finitely generated projectives coincide in the category of modules over a commutative ring.*

Let $d\mathscr{C}$ be the full subcategory of dualizable objects in $\mathscr{C}$ and let $k\mathscr{C} \subset d\mathscr{C}$ be the full subcategory of Künneth objects. These are closed symmetric monoidal additive subcategories of $\mathscr{C}$, and D restricts to equivalences of categories $D : d\mathscr{C}^{op} \longrightarrow d\mathscr{C}$ and $D : k\mathscr{C}^{op} \longrightarrow k\mathscr{C}$. Of course, $k\mathscr{M}_R = d\mathscr{M}_R$. From now on, we assume that $d\mathscr{C}$ is skeletally small. Let $\text{Iso}(d\mathscr{C})$ and $\text{Iso}(k\mathscr{C})$ denote the sets of isomorphism classes of objects in $d\mathscr{C}$ and $k\mathscr{C}$; these are both semi-rings under $\vee$ and $\wedge$.

Corollary 1.4. *$\pi_0 : Iso(k\mathscr{C}) \longrightarrow Iso(d\mathscr{M}_R)$ is a monomorphism of semi-rings.*

Proof. Proposition 1.2 (ii) shows that if $\pi_0(X)$ and $\pi_0(Y)$ are isomorphic, then the represented functors $\mathscr{C}(-, X)$ and $\mathscr{C}(-, Y)$ are isomorphic. □

We now impose extra structure on $\mathscr{C}$ which ensures that π_0 is an isomorphism of semi-rings. The idea is to apply the Brown representability theorem [3]. While this may not give maximal generality, we place ourselves in the context of [17] and assume that $\mathscr{C}$ is a stable homotopy category in the sense of [17, 1.1.4]. This amounts to the following conditions. Details may be found in [17], although the notion of compatibility in (b) given there should be replaced by the more structured notion given in the sequel [25].

(a) $\mathscr{C}$ is triangulated and has arbitrary coproducts.
(b) $\mathscr{C}$ is closed symmetric monoidal, compatibly with the triangulation.
(c) $\mathscr{C}$ has a generating set of small dualizable objects.
(d) Every cohomology functor on $\mathscr{C}$ is representable.

Here a "cohomology functor" is an exact additive contravariant functor from $\mathscr{C}$ to Abelian groups that carries coproducts to products.

Proposition 1.5. *If $\mathscr{C}$ is a stable homotopy category, $\pi_0 : Iso(k\mathscr{C}) \longrightarrow Iso(d\mathscr{M}_R)$ is an isomorphism of semi-rings. In particular, π_0 induces an isomorphism of Abelian groups*

$$Pic(k\mathscr{C}) \longrightarrow Pic(d\mathscr{M}_R) = Pic(R).$$

Proof. As one can check directly, if P is a finitely presented R-module, then the functor $(-) \otimes_R P$ commutes with arbitrary products. Of course, a projective R-module P is flat, so that the functor $(-) \otimes_R P$ is exact. Thus, if P is a finitely generated projective R-module, then $(-) \otimes_R P$ is an exact additive functor that carries products to products. The functor π^0 is exact by standard properties of triangulated categories and it carries coproducts to products. Therefore the composite functor on $\mathscr{C}$ that sends an object Y to $\pi^0(Y) \otimes_R P$ is a cohomology functor. It can be represented by an object X, so that $\pi^0(Y) \otimes_R P \cong \mathscr{C}(Y, X)$. Since the action of R on $\pi^0(Y)$ is given by composition of maps in $\mathscr{C}$, this is an isomorphism of R-modules by naturality. In particular, taking $Y = S$, $\pi_0(X) \cong P$. Arguing as in the last step of the proof of Proposition 1.2, we see that X is a retract of some $S^{\vee n}$ and is therefore a Künneth object. This proves that π_0 is an epimorphism. □

Proof of Theorem 0.2. Let $c : \text{Pic}(R) \cong \text{Pic}(k\mathscr{C}) \longrightarrow \text{Pic}(d\mathscr{C}) \equiv \text{Pic}(\mathscr{C})$ be induced by the inclusion $k\mathscr{C} \longrightarrow d\mathscr{C}$. Since the homomorphism of Picard groups associated to any full embedding of symmetric monoidal categories is a monomorphism, c is a monomorphism. Its image consists of the invertible Künneth objects. □

Parenthetically, we relate these Picard groups to the evident groups of units in Grothendieck rings. Let $L(\mathscr{C})$ and $K(\mathscr{C})$ be the Grothendieck rings associated to $\text{Iso}(k\mathscr{C})$ and $\text{Iso}(d\mathscr{C})$. Write $K(R) = K(\mathscr{M}_R)$ and note that $K(R) = L(\mathscr{M}_R)$. The inclusion $k\mathscr{C} \longrightarrow d\mathscr{C}$ induces a homomorphism of rings $L(\mathscr{C}) \longrightarrow K(\mathscr{C})$ and thus a homomorphism of rings $c : K(R) \cong L(\mathscr{C}) \longrightarrow K(\mathscr{C})$. Letting $A^\times$ denote the units of a ring A, we have the commutative diagram

$$\begin{array}{ccc} \mathrm{Pic}(R) \cong \mathrm{Pic}(k\mathscr{C}) & \xrightarrow{c} & \mathrm{Pic}(\mathscr{C}) \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \beta} \\ K(R)^\times \cong L(\mathscr{C})^\times & \xrightarrow[c]{} & K(\mathscr{C})^\times . \end{array} \tag{1.6}$$

The maps β in (1.6) are considered in [24, §3]. The left arrow β is a monomorphism for any R, by [24, 3.8]. We do not know whether or not the bottom arrow c is a monomorphism in general. However, we can prove that this is often the case. Let $G(R)$ denote the Grothendieck group of finitely generated R-modules.

Proposition 1.7. *Let $\mathscr{C}$ be a unital algebraic stable homotopy category. If $\pi_0(X)$ is a finitely generated R-module for all dualizable objects X and the natural map $\iota : K(R) \longrightarrow G(R)$ is a monomorphism, then $c : K(R) \cong L(\mathscr{C}) \longrightarrow K(\mathscr{C})$ is a monomorphism.*

Proof. Let X and Y be Künneth objects of $\mathscr{C}$ such that $X \vee Z \cong Y \vee Z$ for some dualizable object Z. Then $\pi_0(X) \oplus \pi_0(Z) \cong \pi_0(Y) \oplus \pi_0(Z)$ as R-modules. Since ι is a monomorphism, there is a finitely generated projective R-module P such that $\pi_0(X) \oplus P \cong \pi_0(Y) \oplus P$. Let W be a Künneth object such that $\pi_0(W) \cong P$. Then $\pi_0(X \vee W) \cong \pi_0(Y \vee W)$ and thus $X \vee W \cong Y \vee W$ by Corollary 1.4. □

2. Dualizable and invertible G-spectra

To prove Theorem 0.5, we must characterize the invertible G-spectra in terms of G-spaces, and we first characterize the dualizable G-spectra. Here we are comparing the homotopy category $HoG\mathscr{T}$ of based G-spaces to the homotopy category $HoG\mathscr{S}$ of G-spectra, and we may restrict attention to based G-CW complexes and to G-CW spectra. We write Σ^∞ for the suspension G-spectrum functor $HoG\mathscr{T} \longrightarrow HoG\mathscr{S}$.

We write S^V for the one-point compactification of a representation V, by which we understand a finite dimensional real G-inner product space. We continue to write S^V for $\Sigma^\infty S^V$. These linear sphere spectra are invertible elements of $HoG\mathscr{S}$, this being the essential point of the construction of $HoG\mathscr{S}$. We write S^{-V} for the inverse of S^V. We have desuspension functors Σ^{-V} given by smashing with S^{-V}.

Up to equivalence, the finite G-CW spectra are those of the form $\Sigma^{-V}\Sigma^\infty B$ for a finite G-CW complex B and a representation V of G [21, I.8.16]. We have a similar space level characterization of dualizable G-spectra.

Proposition 2.1. *Up to equivalence, the dualizable G-spectra are the G-spectra of the form $\Sigma^{-V}\Sigma^\infty A$, where A is a finitely dominated based G-CW complex and V is a representation of G.*

Proof. By an argument due to Greenlees [23, XVI.7.4], the dualizable G-spectra are the retracts up to homotopy of the finite G-CW spectra, that is, the finitely dominated G-CW spectra. Since the functors $\Sigma^{-V}\Sigma^\infty$ preserve retracts, it is clear that the G-spectra of the statement are dualizable. We must prove conversely that every retract of a finite G-CW spectrum is obtained by applying one of the functors $\Sigma^{-V}\Sigma^\infty$ to a finitely dominated G-CW complex.

Let $X = X_1$ be a retract of a finite G-CW spectrum $\Sigma^{-V}\Sigma^\infty B$, where B is a finite G-CW complex and V is a representation of G. Since $HoG\mathscr{S}$ is triangulated, retracts split. Thus there is a G-spectrum X_2 such that $X_1 \vee X_2 \simeq \Sigma^{-V}\Sigma^\infty B$. Projection and inclusion give idempotent maps $e_i : \Sigma^{-V}\Sigma^\infty B \longrightarrow \Sigma^{-V}\Sigma^\infty B$ such that $e_1 e_2 = 0 = e_2 e_1$ and $e_1 + e_2 = \mathrm{id}$. Explicitly, e_i is the composite

$$\Sigma^{-V}\Sigma^\infty B \simeq X_1 \vee X_2 \xrightarrow{\simeq} X_1 \times X_2 \xrightarrow{\pi_i} X_i \xrightarrow{\iota_i} X_1 \vee X_2 \simeq \Sigma^{-V}\Sigma^\infty B.$$

By the Freudenthal suspension theorem [23, IX.1.4], we can suspend by $V \oplus W$ for W sufficiently large that Σ^∞ gives a bijection from the homotopy classes of self-maps of the G-space $\Sigma^W B$ to the homotopy classes of self-maps of the G-spectrum $\Sigma^\infty\Sigma^W B \cong \Sigma^W\Sigma^\infty B \cong \Sigma^{V\oplus W}\Sigma^{-V}\Sigma^\infty B$. Moreover, we may as well assume that $W \supset \mathbb{R}^2$, so that $\Sigma^W B$ is simply G-connected. Now $\Sigma^{V\oplus W} e_i = \Sigma^\infty f_i$ for idempotent G-maps $f_i : \Sigma^W B \longrightarrow \Sigma^W B$ such that $f_1 f_2 = 0 = f_2 f_1$ and $f_1 + f_2 = \mathrm{id}$. Taking the f_i to be cellular maps, let A_i be the telescope of countably many iterates of f_i. The composite of the pinch map $\Sigma^W B \longrightarrow \Sigma^W B \vee \Sigma^W B$ and the wedge of the canonical maps $\Sigma^W B \longrightarrow A_i$ gives a map $\xi : \Sigma^W B \longrightarrow A_1 \vee A_2$. On passage to fixed points and homology, ξ^H_* realizes the evident isomorphism

$$H_*((\Sigma^W B)^H) \cong f_{1*}H_*((\Sigma^W B)^H) \oplus f_{2*}H_*((\Sigma^W B)^H).$$

Since these fixed point spaces are simply connected, each ξ^H is a weak equivalence and thus ξ is a G-equivalence by the Whitehead theorem. The evident composites

$$\Sigma^{V\oplus W} X_i \longrightarrow \Sigma^W\Sigma^\infty B \cong \Sigma^\infty\Sigma^W B \longrightarrow \Sigma^\infty A_j$$

are 0 if $i \neq j$, and the sum of the composites with $i = j$ is an equivalence. Thus the composite with $i = j = 1$ is an equivalence. This displays X_1 as $\Sigma^{-(V\oplus W)}\Sigma^\infty A_1$, where A_1 is a wedge summand of the finite G-CW complex $\Sigma^W B$. □

We will prove Theorem 0.5 by using the geometric fixed point functors $\Phi^H : HoG\mathscr{S} \longrightarrow Ho\mathscr{S}$ of [21, II§9] to compare invertible G-spectra to invertible spectra. By [21, II.9.9 and II.9.12], for based G-spaces A and for G-spectra X and Y, we have natural equivalences $\Phi^H\Sigma^\infty A \simeq \Sigma^\infty A^H$ and $\Phi^H(X \wedge Y) \simeq \Phi^H(X) \wedge \Phi^H(Y)$. By [21, III.1.9], this implies formally that if X is a dualizable G-spectrum, then $\Phi^H X$ is a dualizable spectrum and $\Phi^H DX \cong D\Phi^H X$. Moreover, by a variant of the Whitehead theorem [23, XVI§6], a map f of G-spectra is an equivalence if and only if each $\Phi^H f$ is an equivalence of spectra.

Recall our notion of a stable homotopy representation from Definition 0.4.

Proof of Theorem 0.5. We must characterize the invertible G-spectra X. Since invertible G-spectra are dualizable, we may assume that $X = \Sigma^{-V}\Sigma^\infty A$, where A is a finitely dominated based G-CW complex and V is a representation of G. By suspending and desuspending by $\mathbb{R}^2$, we may as well assume that A is simply G-connected. By Lemma 1.1, X is invertible if and only if the evaluation map $\varepsilon : DX \wedge X \longrightarrow S_G$ is an equivalence, where S_G is the sphere G-spectrum. This holds if and only if each $\Phi^H\varepsilon$ is a nonequivariant equivalence. By the results cited above, the map $\Phi^H\varepsilon$ is isomorphic to the map $\varepsilon : D\Phi^H(X) \wedge \Phi^H(X) \longrightarrow S$. This map is an equivalence if and only if $\Phi^H(X)$ is an invertible spectrum. By an elementary argument using the Hurewicz and Whitehead theorems (or see [28] or [16]), the only invertible spectra are the spheres $\Sigma^\infty S^n$ for integers n. Since $\Phi^H(\Sigma^{-V}\Sigma^\infty A) \simeq \Sigma^{-V^H}\Sigma^\infty A^H$, X is invertible if and only if, for each H, $\Sigma^\infty A^H \simeq S^{n(H)}$ for some integer $n(H)$. Since we have assumed that A is simply G-connected,

$n(H) \geq 2$. Thus A^H has the same homology as $S^{n(H)}$ and is therefore equivalent to $S^{n(H)}$ by the Hurewicz and Whitehead theorems. We conclude that the G-spectrum X is invertible if and only if the G-space A is a generalized homotopy representation, which means that X is a stable homotopy representation. □

By easy inspections, smash products and duals (= inverses) of stable homotopy representations are stable homotopy representations. This also follows directly from Theorem 0.5.

Corollary 2.2. *$Pic(HoG\mathscr{S})$ is the group of isomorphism classes in $HoG\mathscr{S}$ of stable homotopy representations.*

3. The exact sequence for Pic($HoG\mathscr{S}$)

We prove Theorem 0.1 by combining the formal algebraic considerations of §1, the topological reduction from G-spectra to G-spaces of §2, and a lemma from the work of tom Dieck and Petrie [12, 13] on space level homotopy representations.

Theorem 0.2 applies since $HoG\mathscr{S}$ is a stable homotopy category. Here $R(HoG\mathscr{S})$ is the Burnside ring $A(G)$, which is a well studied ring. In particular, its prime ideals and its localizations at prime ideals are understood [4, 21].

For an invertible G-spectrum $X = \Sigma^{-V}\Sigma^\infty A$, where A is a generalized homotopy representation, let $d_H(X) = n(H) - \dim(V^H)$, where A^H is equivalent to $S^{n(H)}$. Thus $\Phi^H X$ is a sphere spectrum $S^{d(H)}$; $d(H)$ depends only on the conjugacy class (H) of H, and $d_H(X \wedge Y) = d_H(X) + d_H(Y)$ for invertible G-spectra X and Y.

Let $\Psi(G)$ denote the space of conjugacy classes of (closed) subgroups of G. It is a totally disconnected compact metric space [4, 21]. Let $C(G)$ denote the additive group of continuous functions $\Psi(G) \longrightarrow \mathbb{Z}$, where $\mathbb{Z}$ has the discrete topology. It is more usual to restrict attention to subgroups H of finite index in their normalizers, but that is not appropriate for the present purposes.

Definition 3.1. Define the dimension homomorphism $d : Pic(HoG\mathscr{S}) \longrightarrow C(G)$ by letting $d(X) : \Psi(G) \longrightarrow \mathbb{Z}$ send (H) to $d_H(X)$.

We must check that $d(X)$ is continuous. Since our dimensions are defined homotopically rather than geometrically, $d(X)$ depends only on the homotopy type of X and thus only on its isomorphism class in $HoG\mathscr{S}$. By [19, 1.4], a finitely dominated G-CW complex is homotopy equivalent to a G-CW complex that has finitely many orbit types. Now the continuity of $d(X)$ follows from [11, IV.3.4], which describes the behavior of fixed point subspaces with respect to limits of subgroups. We emphasize that the continuity of $d(X)$ is not formal; rather, it depends upon basic facts about the differential topology of compact Lie groups [11, I§5].

As we will discuss in §4, much is known about the image of d, but it is not fully understood. Consider the sequence

$$0 \longrightarrow Pic(A(G)) \xrightarrow{c} Pic(HoG\mathscr{S}) \xrightarrow{d} C(G).$$

We know that c is a monomorphism, hence the following result completes the proof of Theorem 0.1.

Theorem 3.2. *An invertible G-spectrum X is a Künneth object if and only if $d(X) = 0$. Therefore the kernel of d is equal to the image of c.*

Proof. The last clause follows from the definition of the map c in terms of Künneth objects of $HoG\mathscr{S}$. Let $X = \Sigma^{-V}\Sigma^\infty A$ for a representation V of G and a generalized homotopy representation A. As usual, we may as well assume that A is simply G-connected.

Suppose first that X is a Künneth object. Then, by Proposition 1.2(v), X is a retract of $\bigvee_{i=1}^n S_G$ for some n. Suspending by $V \oplus W$ for a sufficiently large W and arguing as in the proof of Proposition 2.1, we find that the G-space $\Sigma^W A$ is a retract of $\bigvee_{i=1}^n S^{V\oplus W}$. Passing to H-fixed point spaces and observing that a sphere that is a retract of a wedge of m-spheres must be an m-sphere, we see that A^H is equivalent to $S^{n(H)}$, where $n(H) = \dim(V^H)$. Thus $d_H(X) = 0$ and X, regarded as an element of $Pic(HoG\mathscr{S})$, is in the kernel of d.

Conversely, suppose that X is in the kernel of d. This means that A^H is equivalent to $S^{n(H)}$, where $n(H) = \dim(V^H)$. Equivalently, $d(\Sigma^\infty A) = d(S^V)$. We must prove that X is a Künneth object. Write $\mathscr{C} = HoG\mathscr{S}$ and identify $A(G)$ with $R(\mathscr{C})$. It suffices to prove that the canonical map

$$\pi^0(Y) \otimes_{A(G)} \pi_0(X) \longrightarrow \mathscr{C}(Y, X)$$

displayed in Proposition 1.2(ii) is an isomorphism for all dualizable G-spectra Y. This holds if, for all maximal ideals q of $A(G)$,

$$\pi^0(Y)_q \otimes_{A(G)_q} \pi_0(X)_q \longrightarrow \mathscr{C}(Y, X)_q \tag{3.3}$$

is an isomorphism. The maximal ideals of $A(G)$ are of the form $q(H, p)$ where p is a prime number and H is a subgroup of G with finite Weyl group $WH = NH/H$ of order prime to p. The ideal $q(H, p)$ consists of all maps $\phi : S_G \longrightarrow S_G$ such that $\deg_H(\phi) \equiv 0 \mod p$, where $\deg_H(\phi)$ is the degree of the fixed point map $f^H : (S^V)^H \longrightarrow (S^V)^H$ of a space level representative $f : S^V \longrightarrow S^V$ of ϕ. We have the following key lemma, which generalizes an observation of tom Dieck and Petrie [12, §2]. We defer its proof to the end of the section.

Lemma 3.4. *Let $X \in Ker(d)$ and let WH have finite order prime to p. Then there are maps $f : S_G \longrightarrow X$ and $k : X \longrightarrow S_G$ such that $deg(f^H) \not\equiv 0 \mod p$ and $deg(k^H) \not\equiv 0 \mod p$.*

Here, since f and k are maps between different objects, the meaning of the relevant "degrees" is not obvious; we will make sense of them below.

Returning to the proof of Theorem 3.2, fix a maximal ideal $q = q(H, p)$ of $A(G)$, and let f and k be as in the lemma. The composite $f \circ k : S_G \longrightarrow S_G$ is a unit in $A(G)_q$ since $\deg((f \circ k)^H) = \deg(f^H)\deg(k^H) \not\equiv 0 \mod p$, so that $f \circ k$ is not in q. Smashing maps $S_G \longrightarrow S_G$ with X gives an isomorphism of rings $\mathscr{C}(S_G, S_G) \cong \mathscr{C}(X, X)$, and $k \circ f$ is a unit in $\mathscr{C}(X, X)_q$. Thus $f_* : \mathscr{C}(S_G, S_G)_q \longrightarrow \mathscr{C}(S_G, X)_q$ is an isomorphism with inverse k_*. Changing back to the notations in (3.3), the vertical arrows in the following naturality diagram are isomorphisms.

$$\begin{array}{ccc}
\pi^0(Y)_q \otimes_{A(G)_q} A(G)_q & \longrightarrow & \pi^0(Y)_q \\
\downarrow{\scriptstyle 1\otimes f_*} & & \downarrow{\scriptstyle f_*} \\
\pi^0(Y)_q \otimes_{A(G)_q} \pi_0(X)_q & \longrightarrow & \mathscr{C}(Y, X)_q
\end{array}$$

Since the top arrow is clearly an isomorphism, so is the bottom arrow. Thus X is a Künneth object and the proof is complete. □

In view of Proposition 1.5 and [24, 2.11], Theorem 3.2 has the following immediate consequence.

Theorem 3.5. *If* X *is a stable homotopy representation such that* $d(X) = 0$*, then* $\pi_0(X)$ *is a finitely generated projective* $A(G)$*-module of rank* 1*.*

Remark 3.6. For finite groups, Theorem 3.2 is a version of [13, 6.5] of tom Dieck and Petrie; for compact Lie groups, it is a version of [10, 1.6] of tom Dieck. Related information about $\mathrm{Pic}(A(G))$ is given in tom Dieck's papers [8, 9]. Theorem 3.5 generalizes [12, Thm.1] of tom Dieck and Petrie, which gives the result for sphere G-spectra S^{W-V}. We have a homomorphism Sph from the real representation ring $RO(G)$, regarded as an abelian group under addition, to $\mathrm{Pic}(\mathrm{Ho}G\mathscr{S})$. It sends $W - V$ to S^{W-V}, and $W - V$ is in the kernel of Sph if and only if S^V is stably G-homotopy equivalent to S^W. A necessary condition for this to hold is that $W - V$ be in the subgroup $RO_0(G)$ of $RO(G)$ generated by those $W - V$ with $\dim V^H = \dim W^H$ for all H. Define $jO(G)$ to be the image of the restriction $Sph : RO_0(G) \longrightarrow \mathrm{Pic}(\mathrm{Ho}G\mathscr{S})$. Clearly $jO(G)$ is contained in the kernel $\mathrm{Pic}(A(G))$ of d. The group $jO(G)$ is studied in [4, 12].

Proof of Lemma 3.4. We first observe that we need only construct $f : S_G \longrightarrow X$. Indeed, if we can do this, then we can construct $f' : S_G \longrightarrow DX$ in the same fashion. Taking the smash product of f' with the identity map of X and composing with the equivalence $\varepsilon : DX \wedge X \longrightarrow S_G$, we obtain the desired map $k : X \longrightarrow S_G$.

Suspending maps $S_G \longrightarrow X$ by a sufficiently large representation $V \oplus W$, we reduce the problem to consideration of space level maps $S^{V\oplus W} \longrightarrow \Sigma^W A$. Changing notations, it suffices to consider maps $S^V \longrightarrow A$, where V is a representation and A is a generalized homotopy representation such that $A^H \simeq S^{V^H}$ for all $H \subset G$. We may as well assume that A and S^V are G-simply connected, so that $n(H) \equiv \dim(V^H) \geq 2$ for all H.

We must make sense of $\deg(f^H)$ for a G-map $f : S^V \longrightarrow A$. There is a standard way of doing this, due to Laitinen [20, §2] and discussed in detail by tom Dieck [11, pp 169-173]. Tom Dieck considers maps between generalized homotopy representations A and B with the same dimension functions $\{n(H)\}$, and he assumes that the fixed point spaces A^H and B^H both have topological dimension $n(H)$. However, the use of this hypothesis is to obtain restrictions on the dimensions $\{n(H)\}$, and it therefore suffices to assume that either A or B has this property. Since S^V has this property, the discussion applies in our situation. The conclusion is that S^V and A have the same orientation behavior and admit coherent choices of fundamental classes in the homologies of their fixed point spaces. Use of these fundamental classes fixes the degrees $\{\deg(f^H)\}$.

An elementary obstruction theory argument shows that we can extend a nonequivariant map $S^{V^H} \longrightarrow A^H$ of degree one to an H-map $e : S^V \longrightarrow A$; see e.g. [11, II.4.11(ii)]. To obtain the required G-map $f : S^V \longrightarrow A$, we apply a transfer argument. Suspending further if necessary, we can assume that G/H embeds as a sub G-space of V. Using a tubular neighborhood of the embedding and the Pontryagin-Thom construction, we obtain a G-map $S^V \longrightarrow G_+ \wedge_H S^W$, where W is the complement in V of the tangent space of G/H at eH (see e.g. [21, II.5.1]). Using the inclusion $W \subset V$ there results a G-map $t : S^V \longrightarrow G_+ \wedge_H S^V$. We define f to be the composite

$$S^V \xrightarrow{t} G_+ \wedge_H S^V \xrightarrow{\mathrm{id} \wedge_H e} G_+ \wedge_H A \xrightarrow{\xi} A,$$

where ξ is given by the action of G on A. The $W(H)$-space $(G_+ \wedge_H A)^H$ is the wedge of $|W(H)|$ copies of A, with $W(H)$ permuting the wedge summands, and similarly with A replaced by S^V. By virtue of our coherent choices of orientations, we see that f^H is the sum of $|W(H)|$ homeomorphisms, each of which has degree 1. Thus the degree of f^H is $|W(H)|$, which is prime to p. □

4. Remarks on homotopy representations

Since our definition of a generalized homotopy representation differs slightly from the usual one, we give a comparison. In the literature, homotopy representations are defined as unbased spaces, and joins are used instead of smash products. We shall reinterpret the classical definitions in the based context appropriate to stable homotopy theory, and we require A^H to have the homotopy type of a sphere $S^{n(H)}$ for each (closed) subgroup H of G.

With this understanding, tom Dieck's definition [11, II.10.1] of a generalized homotopy representation A replaces our condition that the G-CW complex A be finitely dominated by the conditions that A have finite dimension and finitely many orbit types. We are interested in G-homotopy types, and our definition, unlike tom Dieck's, is homotopy invariant. We have the following comparison.

Proposition 4.1. *Let A be a G-CW complex such that A^H has the homotopy type of a sphere $S^{n(H)}$ for each $H \subset G$. If A is finitely dominated, then A is homotopy equivalent to a finite dimensional G-CW complex B having finitely many orbit types. Conversely, if A is finite dimensional and has finitely many orbit types, then A is finitely dominated.*

Proof. By [19, Thm D] or [22, 14.9], if A is finitely dominated, then it is homotopy equivalent to a finite dimensional G-CW complex A'. Then, by [19, 1.4], A' is homotopy equivalent to a G-CW complex B having finitely many orbit types. With the proof of the cited result, B is still finite dimensional. The converse is proven (although only stated for actual homotopy representations) by Lück [22, 20.2]. □

Thus our definition of a generalized homotopy representation is just a homotopy invariant modification of the usual one.

Homotopy representations are restricted kinds of generalized homotopy representations. The crucial restriction is the requirement that A^H be an $n(H)$-dimensional space, and that is required in all definitions in the literature. This restriction gives control on the possible values taken by the image of d, as we used implicitly in the obstruction theory step of the proof of Lemma 3.4. In [11, II.10.1], but not in [13] and most other sources, two further restrictions are required on A for it to qualify as a homotopy representation, namely

(i) The set $\mathrm{Iso}(A)$ of isotropy groups of A is closed under intersection.
(ii) If $H \in \mathrm{Iso}(A)$ is a proper subgroup of K, then $n(H) > n(K)$.

Observe that both conditions can be arranged by smashing A with S^V for a well chosen representation V. Thus, for stable purposes, we can assume these conditions without loss of generality. We have the following comparison.

Proposition 4.2. *Let G be finite or a torus. For any generalized homotopy representation A, there is a representation V such that $A \wedge S^V$ is equivalent to a homotopy representation B. Therefore every element of $Pic(HoG\mathscr{S})$ can be represented as $\Sigma^{-W}\Sigma^\infty B$ for some homotopy representation B and representation W.*

Proof. First assume that G is finite. Under conditions on A specified in [13, 6.1], [13, 6.6] proves that A is equivalent to a homotopy representation. When each A^H is 2-connected, as can be arranged by smashing with S^3, the conditions are versions of (i) and (ii) above, and they can be arranged by smashing with a suitable S^V.

When G is a torus, the result is proven in [7, p. 463], where it is shown that V and W can be found such that $A \wedge S^V$ is equivalent to S^W. □

The following definition and results help to compare our work with the literature.

Definition 4.3. Define $V(G)$ to be the Grothendieck group associated to the monoid $M(G)$ under smash product of equivalence classes of homotopy representations, with $[S^0]$ as unit. Note that $[A] = [A']$ in $V(G)$ if and only if $A \wedge B$ is equivalent to $A' \wedge B$ for some B. An isomorphic group is obtained using unbased homotopy representations and the join operation. Define $V'(G)$ similarly, but using generalized homotopy representations.

In these groups, inverses are adjoined formally, whereas actual inverse topological objects are present in $\mathrm{Pic}(HoG\mathscr{S})$. Proposition 4.2 implies the following result.

Corollary 4.4. *If G is finite or a torus, the canonical map $V(G) \longrightarrow V'(G)$ is an isomorphism.*

As far as we know, there is no information in the literature about the relationship between homotopy representations and generalized homotopy representations for more general compact Lie groups. It is natural to hope for the following conjecture.

Conjecture 4.5. *The canonical map $V(G) \longrightarrow V'(G)$ is an isomorphism for any compact Lie group G.*

Proposition 4.6. *There is a canonical isomorphism $V'(G) \longrightarrow Pic(HoG\mathscr{S})$.*

Proof. The functor Σ^∞ gives a map of monoids $M(G) \longrightarrow \mathrm{Pic}(HoG\mathscr{S})$, which extends uniquely to a map of groups $V(G) \longrightarrow \mathrm{Pic}(HoG\mathscr{S})$. This map is an epimorphism by Theorem 0.5. It is a monomorphism since if A and B are generalized homotopy representations such that $\Sigma^\infty A$ is equivalent to $\Sigma^\infty B$, then there is a representation V such that $A \wedge S^V$ is equivalent to $B \wedge S^V$. □

The image of $d : V(G) \longrightarrow C(G)$ has been studied extensively; see [6, 7, 11, 13, 14, 15] for finite groups and [1, 2] for general compact Lie groups. It is rarely an epimorphism, although it is so trivially if G is cyclic of order 2. For finite nilpotent groups G, in particular for p-groups, the image of d is realized by linear representations. In more detail, there are necessary conditions, called the Borel-Smith conditions, for an element $f \in C(G)$ to be the dimension function of a homotopy representation, and when G is nilpotent every such f is $d(S^\alpha)$ for some virtual representation α; see [11, III§5]. In [2], Bauer defines a subgroup $D(G)$ of $C(G)$ and displays a short exact sequence

$$0 \longrightarrow \mathrm{Pic}(A(G)) \longrightarrow V(G) \longrightarrow D(G) \longrightarrow 0.$$

It refines the exact sequence of Theorem 0.1 to a short exact sequence when G is finite or a torus, and this will remain true for general compact Lie groups G if Conjecture 4.5 holds.

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